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Hypothesis of Bosonic Antimatter

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A study based on the hypothesis that bosonic commuting fields annihilate fermionic anticommuting fields is presented. If this process is allowed, scalar grading of the $SO(1;3)$ group can be employed and a supersymmetric superinvariant quantum electro-dynamics (QED) can be developed. Further, the Pauli spin–statistics theorem does not hold, the gauge invariance holds for kinetic terms only, and scattering unitarity is preserved.

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Four prominent problems persist in modern physics. First, the elegant theory of supersymmetric fields[1, 2] is not supported by any experimental evidence, following analysis of a dataset of 35 fb^{-1}.[3] Second, antimatter statistics should be fermionic according to the Pauli spin–statistics theorem. However, the supporting experimental evidence is very limited: no positron scattering has been measured,[4] and no antiatoms or antimolecules with two positrons have been observed.[5, 6] The ground-state spin configuration of a dipositronium molecule is inaccessible because of the short lifetime of dipositronium in a magnetic field.[4] The third problem is related to the natural proof of the Pauli spin–statistics theorem.[7] The exchange symmetry of the $SO(1;3)$ many-body wave function computed through group embedding into $SO(N;3N)$ space[8, 9] yields anticommutation of left spinors and commutation of all other pairs of spinor fields.

Fourth, the universe has very clear matter–antimatter asymmetry, which is understood today in terms of sophisticated symmetry violations[10, 11] that are unlikely to occur[12]. The development of modern physics clearly collided with the barrier of matter–antimatter asymmetry 50 years ago, with the report by Sakharov[13]. To overcome this barrier, one of the following avenues should be pursued: development of new mechanisms of symmetry breaking satisfying Sakharov's conditions or adjustment of the well-established quantum electro-dynamics (QED) formalism, with recalculation of some basic QED predictions.

By adopting the second approach, a formalism yielding a hypothesis of bosonic antimatter can be obtained. Therefore, with the required additional caution, a supersymmetric superinvariant QED action is proposed in this study. It is speculated that matter and antimatter have opposite observable statistics, which explains matter–antimatter asymmetry. As matter is known to annihilate antimatter, the present theory ultimately necessitates the possibility that a fermionic field can be annihilated by a bosonic field, and vice versa.

At first glance, a symmetric many-body wave cannot annihilate an antisymmetric many-body wave. However, particle-antiparticle annihilation is a single-body process and can apply to fields with opposite statistics. Note that, if N fermionic particles meet N bosonic antiparticles, they will not simultaneously annihilate (this process is forbidden). Instead, annihilation will occur on a one-by-one basis.

The obvious difficulty associated with a theory involving anticommuting left spinors and commuting right spinors is that the low-energy fermions of modern QED are in a mixed state of left and right spinors; thus, they cannot be half commuting and half anticommuting. Therefore, a low-energy matter field must be constructed from the left and left-conjugated spinors, while antimatter can be generated from the right and right-conjugated spinors.

The derivation of this theory is initiated using a superalgebra[14] obtained by utilizing an orthogonal Lie algebra and a scalar supercharge. This yields the same symmetry (spin) for both commuting and anticommuting fields, as in the theory of disordered metals.[15] Commuting fields in the theory of metals are replicas of anticommuting fields, yielding a few subalgebras of $\mathfrak{gl}(n|n)$.[16] In the present theory, the supercharge acts as the charge conjugation. Thus, anticommuting and commuting fields have the same symmetry (spin) but represent physically different particles.

The spatial rotations for a bispinor representation ψ of the $SO(1;3)$ group are generated by the matrix T, where

$$\begin{aligned} &\delta\psi = T\psi \ , \quad \delta\psi^c = T\psi^c \ , \quad CT^*C = T \\ &\delta\psi^u = -\psi^u T \ , \quad \delta\psi^{cu} = -\psi^{cu} T \ , \quad C'T^tC' = -T \\ &\psi^c = C\psi^* \ , \quad \psi^u = \psi^t C' \end{aligned} \tag{1}$$

Here, superscript t represents a transposed vector, C is the charge conjugation matrix, and C' is the time inversion matrix. Explicitly, $C^2 = 1$, $C^t = C$ and $C'^2 = 1$ depend on the choice of representation for T.

Superrotations are generated by the Grassmannian supercharge Q satisfying $QQ^c = -Q^cQ$, such that

$$\begin{aligned} &\delta_q\psi = Q^c\psi^c \ , \quad \delta_q\psi^c = Q\psi \ , \quad Q^c = CQC \\ &\delta_q\psi^u = \psi^{cu}Q^c \ , \quad \delta_q\psi^{cu} = \psi^u Q \ , \end{aligned} \tag{2}$$

Here, ψ is the $SO(1;3)$ bispinor $\psi_L \oplus \psi_R$ and ψ^c is the charge-conjugated bispinor $\psi^*_R \oplus \psi^*_L$. The ψ_L and ψ_R fields denote left and right spinors, respectively (dotted and undotted spinors in other notation systems). Therefore, the commutation rules are as follows:

$$\begin{aligned} &[\![\psi_\alpha, \phi_\beta]\!] = 0 \;, \quad [\![\psi^*_\alpha, \phi^*_\beta]\!] = 0 \\ \Leftrightarrow\;& \{\psi_L, \phi_L\} = 0 \;, \quad [\psi_R, \phi_R] = 0 \;, \quad [\psi_L, \phi_R] = 0 \;, \end{aligned} \tag{3}$$

where ϕ is another bispinor. In this multiplet, the ψ_R and ψ^*_R fields and the ψ_L and ψ^*_L fields are commuting and anticommuting components, respectively.

The superalgebra is constructed utilizing the second variations in Eqs. (1) and (2). The grading is defined as follows:

$$\begin{aligned} &L_0 = \mathrm{End}(T) \quad L_{-1} = \mathrm{End}(Q^c) \quad L_1 = \mathrm{End}(Q) \\ &\mathfrak{sso}(1;3|1) = L_{-1} \oplus L_0 \oplus L_1 \;. \end{aligned} \tag{4}$$

Here, supercharges Q and Q^c must anticommute to an ideal of L_0, which is zero for the case of L_0, being the special orthogonal algebra.[17]

The gauge invariance is one of the most essential components of QED. In the present theory, the gauge invariance is introduced as

$$\delta_a \psi = i\Lambda(x)\gamma^{\text{FIVE}}\psi \;, \quad \delta_a \psi^c = i\Lambda(x)\gamma^{\text{FIVE}}\psi^c. \tag{5}$$

Here, $\Lambda(x)$ is the wave-function phase and the γ matrices can be defined as in Refs. [8] or [18, Ch. II].

For the generic superinvariant Lorenz vector v^μ and superinvariant Lorenz scalar s (note: the scalar below is not gauge invariant, similar to the theory of Majorana fermions), the following relations hold:

$$\begin{aligned} &v^\mu = \phi^u\gamma^\mu\psi - \phi^{cu}\gamma^\mu\psi^c \;, \quad \delta_q v^\mu = \delta_a v^\mu = 0, \\ &\delta v^\mu = \phi^u[\gamma^\mu, T]\psi - \phi^{cu}[\gamma^\mu, T]\psi^c, \\ &s = \phi^{cu}\psi - \phi^u\psi^c \;, \quad \delta s = \delta_q s = 0. \end{aligned} \tag{6}$$

Hence, the superinvariant QED action

$$\begin{aligned} \mathcal{S} &= i\sum_{p_0>0,\vec{p}} \Big\{ \psi^u_{-p}\gamma^\mu p_\mu \psi_p - m\psi^u_{-p}\psi^c_p \\ &\qquad - \psi^{cu}_{-p}\gamma^\mu p_\mu \psi^c_p + m\psi^{cu}_{-p}\psi_p \Big\} \\ S_I &= i\frac{e}{2}\sum_{pp'} \Big\{ \psi^u_{-p'}\tilde{\gamma}^\mu A_{\mu p'-p}\psi_p - \psi^{cu}_{-p'}\tilde{\gamma}^\mu A_{\mu p'-p}\psi^c_p \Big\} \end{aligned} \tag{7}$$

is derived, where e and m are the particle charge and mass, respectively, $\tilde{\gamma}^\mu \equiv \gamma^\mu\gamma^{\text{FIVE}}$, and the $C'\tilde{\gamma}^\mu = C'\gamma^\mu\gamma^{\text{FIVE}}$ matrices are all symmetric. The factor 1/2 in the interaction part of the action S_I compensates for double counting. For the same reason, S is expressed as a sum over $p_0 > 0$. The Fourier–transformed fields are defined such that $\psi^c_p \equiv (\psi_{-p})^c$.

The commuting and anticommuting fields are separated using the following component re-ordering:

model	4-vector form	scalar form	gauge invariant	super invariant
Textbook [18, Ch. II]	$\phi^*_L\psi_L + \phi^*_R\psi_R$	$\phi^*_L\psi_R + \phi^*_R\psi_L$	yes	no
Majorana	$\phi^*_L\psi_L$	$\phi^*_L\psi^*_L + \phi_L\psi_L$	vector only	no
This work	$\phi_L\psi_R + \phi_R\psi_L$ + c.c.	$\phi^*_L\psi_R + \phi^*_R\psi_L$ + c.c.	or vector or scalar	yes

TABLE I: Comparison of symmetries from various theories. Here, γ matrices and other spin-dependent factors are omitted.

$$\begin{matrix}\psi_p \\ \psi^c_p\end{matrix} = \begin{matrix}\psi_{R\uparrow}\\ \psi_{L\downarrow}\\ \psi_{L\uparrow}\\ \psi_{R\downarrow}\\ \psi^c_{R\uparrow}\\ \psi^c_{L\downarrow}\\ \psi^c_{L\uparrow}\\ \psi^c_{R\downarrow}\end{matrix} = \begin{matrix}\psi_{R\uparrow}\\ \psi_{L\downarrow}\\ \psi_{L\uparrow}\\ \psi_{R\downarrow}\\ \psi^*_{L\downarrow}\\ \psi^*_{R\uparrow}\\ \psi^*_{R\downarrow}\\ \psi^*_{L\uparrow}\end{matrix}, \quad \begin{matrix}\Psi^R_p \\ \Psi^L_p\end{matrix} = \begin{matrix}-\psi_{R\uparrow}\\ \psi^*_{R\downarrow}\\ -\psi^*_{R\uparrow}\\ \psi_{R\downarrow}\\ \psi^*_{L\uparrow}\\ \psi_{L\downarrow}\\ \psi_{L\uparrow}\\ \psi^*_{L\downarrow}\end{matrix}. \tag{8}$$

Hence, the supersymmetric QED action is

$$\begin{aligned} \mathcal{S} &= i\sum_{p_0>0,\vec{p}} \Big\{ \Psi^R_{-p}(\gamma^\mu p_\mu - m)\Psi^L_p \\ &\qquad + \Psi^L_{-p}(-\gamma^{\mu t}p_\mu - m)\Psi^R_p \Big\} \\ \mathcal{S}_I &= ie\sum_{pp'} \Psi^R_{-p'}\tilde{\gamma}^\mu A_{\mu p'-p}\Psi^L_p \end{aligned} \tag{9}$$

The "antiparticle" second term of the action S can be combined with the "particle" first term via the substitution $p_0 \to -p_0$. The sum in the exponent e^{S+S_I} should be taken as a product of exponents, to handle the lack of commutation between Ψ^L_p.

The theory features two Green functions that can be computed through regularization of superintegrals over commuting $\Psi_{\pm pR}$ and anticommuting $\Psi_{\pm pL}$ variables. Thus,

$$G^{LR}_p = i\left\langle \Psi^L_p \Psi^R_{-p}\right\rangle = \frac{\gamma^\mu p_\mu + m}{p_0^2 - \vec{p}^2 - m^2} \tag{10}$$

$$G^{RL}_p = i\left\langle \Psi^R_p \Psi^L_{-p}\right\rangle = \frac{-\gamma^{\mu t} p_\mu + m}{p_0^2 - \vec{p}^2 - m^2}\;. \tag{11}$$

Here, $p_0 > 0$, and G^{LR}_p and G^{RL}_p propagate particles and antiparticles, respectively. The regularization requires additional terms such as $(i\xi - 0)\Psi^R\Psi^{R*}$ in the action, where ξ is a non-vanishing real number; then, ξ is dropped from the final result.

Particles are created with the fermionic operator Ψ^L_p and annihilated with the bosonic operator Ψ^R_{-p}. Further, antiparticles are created with the bosonic operator Ψ^R_p and annihilated with the fermionic operator Ψ^L_{-p}. Speculatively, it is claimed that the experimentally measured statistics of a particle are determined upon its birth.

The scalar mass term in the action described by Eqs. (7) and (9) is similar to textbook QED.[18, Ch. II] However, the vector kinetic term is different (see comparison in Table I). This is the only method of expressing a supersymmetric Lorentz invariant action found by the author. The gauge invariance must also be incorporated; however, v^μ and s cannot be gauge invariant at the same time.

The generic form of the conserved current derived from any of the kinetic terms in Eq. (7) utilizing the Noether theorem is $j^\mu = \psi^{cu}_{-p}\gamma^\mu\delta_x\psi^c_p$, where δ_x represents any of the variations introduced in Eq. (1). With $\delta_x = \delta_q$, we obtain $j^\mu = a\psi^\dagger\gamma^0\gamma^\mu\psi + b\psi^\dagger\gamma^{\text{FIVE}}\gamma^0\gamma^\mu\psi$, where a, b are arbitrary numbers. Here, j^μ is Lorentz and gauge invariant; however, it is not superinvariant. In addition, j^μ is a bosonic 4-vector, because it pairs a left-conjugated spinor with a left spinor (both anticommuting) and a right-conjugated spinor with a right spinor (both commuting). Therefore, the bosonic spin-one field may be regarded as being composed of two spin-half fields, even under the theory of mixed statistics for spin-half fields. Note that a comment in Ref. [19, §25] raises concern regarding a possible fermionic spin-one field violating the Pauli spin–statistics theorem if mixed statistics for spin-half fields is allowed.

Examples of four main interaction processes are given to illustrate supermultiplet propagation. Four corrections to the two-body propagator are made by integrating out the vector potential fields $A_{\mu q}$. One has

$$\mathcal{S}_I = ie^2 \sum_{p-p'+k-k'=0} D_{p'-p}\Psi^R_{-p'}\tilde{\gamma}^\mu\Psi^L_p\Psi^R_{-k'}\tilde{\gamma}_\mu\Psi^L_k \quad (12)$$

where $D_p = i\langle A_p A_{-p}\rangle$ is the photonic propagator and the scattering channels are

$$\begin{array}{rl} p_0, p_0', k_0, k_0' > 0 & \text{particle – particle} \\ p_0, p_0', k_0, k_0' < 0 & \text{antiparticle – antiparticle} \\ p_0, p_0' > 0,\ k_0, k_0' < 0 & \text{particle – antiparticle} \\ p_0, k_0' < 0,\ p_0', k_0 > 0 & \text{annihilation and birth .} \end{array}$$

The scattering unitarity [19, §72] for unsymmetrized waves means that

$$M_{fi} = M^*_{if},\ M_{fi'} = M^*_{i'f},\ M_{f'i} = M^*_{if'}\ , M_{f'i'} = M^*_{i'f'} \quad (13)$$

where M is the scattering amplitude for a two–body initial state i, a two-body final state f, and states with permuted particles i' and f'. The scattering unitarity is evidently preserved upon symmetrization or alternation over the initial or final states.

This theory can be further developed using various approaches. The procedure depends on the definition of the observed current and the density and formalism used to calculate this current. This is the point at which the collateral provided by experimental data is ultimately necessary. Let us assume that experimental data confirm that antiparticle scattering differs from particle scattering. This can be explained by symmetrization (alternation) of the amplitude of Eq. (12) over initial states only. However, justification of this procedure requires considerable effort, for example, through application of the Keldysh formalism.[20]

The concept of graded superalgebra $\mathfrak{sso}(1;3|1)$ originates from exchange rotations derived using the group embedding method. The scalar grading of the orthogonal algebra $so(1;3)$ can be generalized for higher-dimensional algebras $so(N;3N)$. The branching rules of $4N$-dimensional spinors upon group embedding of

$$so(N+M;3N+3M) \to so(N;3N) \otimes so(M;3M) \quad (14)$$

are[21, 22]

$$\begin{array}{l} \psi_L^{N+M} \to \psi_L^N \otimes \psi_R^M \oplus \psi_R^N \otimes \psi_L^M \\ \psi_R^{N+M} \to \psi_L^N \otimes \psi_L^M \oplus \psi_R^N \otimes \psi_R^M\ . \end{array} \quad (15)$$

It is, therefore, possible to define ψ_L and ψ_R respectively as anticommuting and commuting fields for any N, and these commutation properties are preserved upon embedding of the maximal subalgebra given by Eq. (14).

For odd N, the superalgebra $\mathfrak{sso}(N;3N|1)$ is obtained by grading of $so(N;3N)$ with a scalar supercharge Q^N. The variation of Eq. (15) allows to derive branching rules for $so(N;3N)$ generators T^N and the scalar supercharge Q^N. The branching rules for T^N follow theorem stating that the adjoint representation is branched to sum of adjoint representations and product of the vector representations of subalgebras. The branching rules for Q^N are

$$\begin{array}{l} Q^{N+M} = Q^N \otimes 1^M \quad \text{odd } N, N+M\ . \\ Q^{N+M} = 0 \quad \text{odd } N, M \end{array} \quad (16)$$

The scalar supercharge must be zero for even N, because in this case ψ^N is self-conjugated. Therefore, it is not possible to define grading for $so(N;3N)$ with even N. Use of the $\mathfrak{sso}(N;3N|1)$ notation for even N remains possible, assuming this is the same algebra $so(N;3N)$ with anticommuting left and commuting right spinors.

The following maximal supersubalgebra embedding segregates the $4N$-dimensional space to 4-dimensional subspaces:

$$\mathfrak{sso}(N;3N|1) \to \mathfrak{sso}(1;3|1) \otimes \cdots \otimes \mathfrak{sso}(1;3|1) \quad (17)$$

The exchange rotation is the transformation exchanging any two 4-dimensional subspaces of $4N$-dimensional space. A tedious but straightforward calculation yields the following transformation of the $\mathfrak{sso}(N;3N|1)$ spinor:

$$\begin{array}{c} \psi_{LL}(x,x') \\ \psi_{LR}(x,x') \\ \psi_{RL}(x,x') \\ \psi_{RR}(x,x') \end{array} \to \begin{array}{c} -\psi_{LL}(x',x) \\ \psi_{RL}(x',x) \\ \psi_{LR}(x',x) \\ \psi_{RR}(x',x) \end{array}, \quad (18)$$

where indexes and coordinates correspond to the two exchanged subspaces.[8] The 2^{2N} components of the $\mathfrak{sso}(N;3N|1)$ spinor are indexed according to the indexes of the N embedded $\mathfrak{sso}(1;3|1)$ 4-component bispinors.

This symmetry lowering to four dimensions gives the generalization of the Heisenberg spin exchange matrix:

$$\begin{array}{ccc} \psi_L(x)\psi_L(x') & & -\psi_L(x')\psi_L(x) \\ \psi_L(x)\psi_R(x') & \rightarrow & \psi_R(x')\psi_L(x) \\ \psi_R(x)\psi_L(x') & & \psi_L(x')\psi_R(x) \\ \psi_R(x)\psi_R(x') & & \psi_R(x')\psi_R(x) \end{array} . \tag{19}$$

Therefore, it can be stated that the commutation rules of $\mathfrak{sso}(1;3|1)$ operator fields Eq. (19) are in agreement with the exchange rotations Eq. (18) of the $\mathfrak{sso}(N;3N|1)$ field.

In summary, a hypothesis of bosonic antimatter based on supersymmetric, Lorentz invariant QED has been proposed in this paper. In this theory, the superfields are associated with representations of $\mathfrak{sso}(1;3|1)$ algebra with scalar grading. The branching rules for $\mathfrak{sso}(N;3N|1)$ graded algebras to $\mathfrak{sso}(1;3|1)$ embedded supersubalgebras are discussed. In addition, it is demonstrated that sub-space exchange rotations in $\mathfrak{sso}(N;3N|1)$ algebra (as in Eq. (18)) are equivalent to field permutation Eq. (19) in graded $\mathfrak{sso}(1;3|1)$ subalgebra.

The theory is feasible only if annihilation between commuting and anticommuting fields is permitted in nature. This is the major difference between proposed supersymmetric QED and the textbook theory. The energy dispersion of both matter and antimatter particles is preserved together with the propagators and scattering amplitude. Therefore, diagrammatic calculation of most observable processes should reproduce the well-established results. The difference appears in summation of diagrams over permutation of particles and loop diagrams.

The author believes that the proposed theory can explain matter–antimatter asymmetry of the universe due to a lack of degeneracy pressure in antimatter stars. In this way, this problem will be resolved without charge, parity, and time reversal (CPT) violations, as opposed to some modern theories.[11] The author also hopes that this publication will motivate research on antimatter–antimatter interaction, as almost no experimental data are available at present.

[1] L. Corwin, Y. Ne'eman, and S. Sternberg, Rev. Mod. Phys. **47**, 573 (1975), URL `https://link.aps.org/doi/10.1103/RevModPhys.47.573`.

[2] M. F. Sohnius, Physics Reports **128**, 39 (1985), ISSN 0370-1573, URL `http://www.sciencedirect.com/science/article/pii/0370157385900237`.

[3] A. Petridis, in *52nd Rencontres de Moriond EW 2017* (2017), URL `https://indico.in2p3.fr/event/13763/contributions/15200/attachments/12703/15595/3_AndreasPetredis.pdf`.

[4] J. R. Danielson, D. H. E. Dubin, R. G. Greaves, and C. M. Surko, Rev. Mod. Phys. **87**, 247 (2015), URL `https://link.aps.org/doi/10.1103/RevModPhys.87.247`.

[5] J. Adam (ALICE Collaboration), Phys. Rev. C **93**, 024917 (2016), URL `https://link.aps.org/doi/10.1103/PhysRevC.93.024917`.

[6] K. P. Jungmann, Nature **524**, 168 (2015), URL `http://www.nature.com/nature/journal/v524/n7564/abs/524168a.html`.

[7] M. V. Berry, Nonlinearity **21**, T19 (2008), URL `http://stacks.iop.org/0951-7715/21/i=2/a=T02`.

[8] D. L. Miller (2016), manuscript submitted for publication.

[9] D. L. Miller (2017), manuscript submitted for publication.

[10] M. Dine and A. Kusenko, Rev. Mod. Phys. **76**, 1 (2003), URL `https://link.aps.org/doi/10.1103/RevModPhys.76.1`.

[11] V. A. Kostelecký and N. Russell, Rev. Mod. Phys. **83**, 11 (2011), URL `https://link.aps.org/doi/10.1103/RevModPhys.83.11`.

[12] K. Abe (T2K Collaboration), Phys. Rev. D **95**, 111101 (2017), URL `https://link.aps.org/doi/10.1103/PhysRevD.95.111101`.

[13] A. D. Sakharov, Pisma Zh. Eksp. Teor. Fiz. **5**, 32 (1967), [Usp. Fiz. Nauk161,61(1991)], URL `http://stacks.iop.org/0038-5670/34/i=5/a=A08`.

[14] V. G. Kac, Advances in Mathematics **26**, 8 (1977), ISSN 0001-8708, URL `http://www.sciencedirect.com/science/article/pii/0001870877900172`.

[15] K. B. Efetov, Advances in Physics **32**, 53 (1983), URL `http://dx.doi.org/10.1080/00018738300101531`.

[16] A. Kirillov, F. Berezin, J. Niederle, and R. Kotecký, *Introduction to Superanalysis*, Mathematical Physics and Applied Mathematics (Springer Netherlands, 2013), ISBN 9789401719636, URL `https://books.google.co.il/books?id=srHsCAAAQBAJ`.

[17] F. A. Berezin and V. S. Retah, Funct. Anal. Appl. **12-1**, 48 (1978).

[18] A. Zee, *Quantum Field Theory in a Nutshell*, Nutshell handbook (Princeton Univ. Press, Princeton, NJ, 2003), URL `https://cds.cern.ch/record/706825`.

[19] V. Berestetskii, E. Lifshitz, and Pitaevskiĭ, *Quantum Electrodynamics* (1982).

[20] L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964), [Sov. Phys. JETP20,1018(1965)], URL `http://www.jetp.ac.ru/cgi-bin/e/index/e/20/4/p1018?a=list`.

[21] R. Slansky, Physics Reports **79**, 1 (1981), ISSN 0370-1573, URL `http://www.sciencedirect.com/science/article/pii/0370157381900922`.

[22] SAGE, *Maximal subgroups and branching rules* (2017), URL `http://doc.sagemath.org/html/en/thematic_tutorials/lie/branching_rules.html`.