Proof of anticommutation between exchange and charge conjugation

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We prove anticommutation between the exchange and charge conjugation of Lorenz invariant bispinors by raising the Lorenz symmetry to $SO(3N, N)$ and lowering it back to $SO(3,1)$. This finding contradicts one of the foundations of the spin–statistics theorem and the exclusion principle for antimatter.

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In the 2008 review paper[1], Prof. M. Berry called for elementary and natural understanding of the relationship between spin and statistics. This is performed in the present paper by raising and lowering the spacial symmetry of the Dirac's bispinor. The high-symmetry state is hypothetical, which allows us to replace the discrete particle exchange symmetry with a continuous rotational symmetry. This method has been applied to few symmetries of quantum fields, however, without definitive conclusions.[1]

In quantum electrodynamics, the many-body field propagation is described by the $2N$-leg Greens function of $2 \times 4N$ coordinates $x_j^\mu, x_j^{\mu\prime}$, where $\mu = 0,1,2,3$ and $j = 1 \ldots N$ according to the number of particles propagated by the Greens function. The Lorenz symmetry is essentially $SO(3,1)$ rotational symmetry; it simultaneously rotates all 4-vectors x_j, x_j'. The Greens function is a 2^{2N} by 2^{2N} matrix, because it transforms as the direct product of N bispinors.

The exchange operator $\hat{E}$ is also a 2^{2N} by 2^{2N} matrix; its action can be described by the exchange of coordinates

$$\hat{E}_{ij}: \begin{array}{l} x_j^\mu \to x_i^\mu \,, \\ x_i^\mu \to x_j^\mu \,, \\ x_k^\mu \to x_k^\mu \,, \quad k \neq i,j \end{array} \tag{1}$$

the explicit calculation of this matrix is not possible in the framework of the existing theory. It was conjectured by Heisenberg and known as the Heisenberg exchange Hamiltonian. It describes the Pauli exclusion principle and the anticommutation of bispinor quantum fields according to the spin–statistics theorem.

We can explicitly derive the operator $\hat{E}$ by raising symmetry of the $2N$-leg Greens function to $SO(3N, N)$. The transformation Eq. (1) then reduces to rotation in $4N$ dimensional space and can be explicitly calculated in terms of γ-matrices. In particular, it satisfies

$$\hat{E}_{ij}\gamma_j^\mu \hat{E}_{ij} = \gamma_i^\mu \,, \quad \hat{E}_{ij} = \hat{E}_{ji} \,, \quad \hat{E}_{ij}\hat{E}_{ij} = \hat{1} \,, \tag{2}$$

Where the set of $4N$ γ-matrices γ_j^μ represents the $SO(3N, N)$ vector. The dimensionality of the γ-matrices is also 2^{2N} by 2^{2N}, because $SO(3N, N)$ bispinor has 2^{2N} components.[2] Explicit calculation of $\hat{E}$ is straightforward.

The charge conjugation $\hat{C}$, time inversion $\hat{T}$, parity $\hat{P}$, and 4-inversion $\hat{I}^4$ can be generalized to $SO(3N, 3)$ as follow (use the Weyl representation for γ_j^μ)

$$\begin{array}{ll} \hat{C} = \prod_j \hat{C}_j, \; \hat{C}_j \sim \gamma_j^2 & \hat{P} = \prod_j \hat{P}_j, \; \hat{P}_j \sim \gamma_j^0 \\ \hat{T} = \prod_j \hat{T}_j, \; \hat{T}_j \sim \gamma_j^1\gamma_j^3 & \hat{I}^4 = \prod_j \hat{I}_j^4, \; \hat{I}_j^4 \sim \gamma_j^0\gamma_j^1\gamma_j^2\gamma_j^3 \end{array} \tag{3}$$

The anticommutation of the γ-matrices between themselves, along with Eq. (2), gives

$$\begin{array}{ll} \hat{E}\hat{C}\hat{E} = -\hat{C} & \hat{E}\hat{P}\hat{E} = -\hat{P} \\ \hat{E}\hat{T}\hat{E} = \hat{T} & \hat{E}\hat{I}^4\hat{E} = \hat{I}^4 \end{array} \tag{4}$$

and this is the main result of the present work. Upon the symmetry lowering of the $2N$-leg Greens function back to $SO(3,1)$, the commutation relations Eqs. (4) should be preserved, which concludes our proof of anticommutation between the exchange and charge conjugation.

The above result implies that the matter and the antimatter have opposite statistics, or forbidden to have the same statistics. This voids the foundation of the Pauli's proof of the spin–statistics theorem[3]. Therefore, the result does not directly contradict the theorem.

The fermionic (anticommuting) nature of the matter (e.g., electrons) is well-confirmed by the periodic table of elements, direct measurement of electron-electron scattering and other methods. At the same time, any direct probe of antimatter commutation rules is absent. We, therefore, call upon the physics community to directly test the antimatter statistics, e.g., through positron-positron scattering or by examining the spectrum of antihydrogen molecule $\bar{H}_2$.

An experimental confirmation of the present theory will open up a path to solve the paradox of the matter–antimatter asymmetry of the universe in a quantum electrodynamics framework. The antimatter universe will be unstable owing to the lack of degeneracy pressure.[4]

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