Two species Wigner lattice state of unconfined electron-positron plasma and application to the theory of ball lightning

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Thunderstorms generate strong flows of 511 keV gamma rays, indicating that a substantial number of positrons participate in this process. This makes people think that ball lightning is the vortex of electron-positron plasma. The stability of the plasma can be achieved by balancing the attractive Coulomb forces between electrons and positrons by their kinetic energy. The concept of first order phase transition leading to the formation of electron–hole drops in semiconductors is known since the 1970s. However, the electron–hole liquid drop has a short lifetime and cannot describe relatively long-lived ball lightning. Here, I report two–species Wigner crystal solution to the equation of state, which is only possible when positive and negative charges have exactly the same mass. The model is valid for all observed ball lightning temperatures, however at some point it could be quantum melted state. The annihilation of electron–positron pairs and evaporation of electrons and positrons from the crystal surface set the ball lightning lifetime. The evaporation of the crystal is found to be governed by the power law; it is calculated together with the density of the surrounding free electron–positron plasma by matching the chemical potentials. For a realistic temperature range, the density of the crystal is sufficiently low and prevents extensive annihilation of electron–positron pairs. Experimental detection of 511 keV gamma rays sourced from a ball lightning will confirm this theory.

Discovery of 511 keV annihilation radiation coming from thunderstorms [1–6] suggested the presence of positrons in ball lightning [7]. The most comprehensive modern book summarizing ball lightning data [8], Chap. 1, provides the following definition of the ball lightning: *"Ball lightning is a luminous globe that occurs in the course of a thunderstorm. It is most often red, although varying colors including yellow, white, blue, and green have also been often reported for the glowing ball. The size varies widely, but a diameter of one–half of a foot is common. Its appearance is in striking contrast to ordinary lightning, as it often moves in a horizontal path near the earth at a low velocity. It may remain stationary momentarily or change the course while in motion. Unlike the rapid flash of ordinary lightning, ball lightning exists for extended periods of time, several seconds or even minutes."*

A prosperous theory should explain eight characteristics of ball lightning.[9, 10] We will not repeat all of them, particularly challenging facts are (iii) "*its occurrence in both open air and in enclosed spaces such as buildings or aircraft*" and (vii) "*the fact that it appears to pass through small holes, through metal screens, and through glass windows*". Positron based ball lightning has the advantage of high diffusion owing to the small mass of positrons, and according to [7] can penetrate thin layers of solid matter. In this way, the positron based theory explains characteristics (iii) and (vii).

The latest observations and spectroscopy data indicate the presence of soil elements in the ball lightning [11] for the entire lifetime, which one can call fact (ix). A few recent publications have assumed that lightning strikes the ground and evaporates the soil elements from the ball lightning [9]. The lightning strike on a conducting surface produces a significant amount of high–energy electrons [12] and positrons [13] in favor of the electron–positron model.

The stable cloud of electron–positron plasma propagating freely in air is a fascinating new model that lacks rigorous mathematical description. This model is the first example of a stable free low–density plasma and can have a significant impact on the physics of positrons. For instance, very significant experimental and theoretical efforts have been made to build traps for positron and electron–positron plasma [14, 15]; the electron–positron clouds can be seen as natural positron storages.

In this study, I demonstrate the stability of neutral ball lightning made from electron–positron plasma in a wide range of temperatures and independent of air pressure and temperature. So far, it has not been possible to find a satisfactory stable electron–ion plasma model for ball lightning (please refer [8], Chap. 8L). The calculation shows that the electron exchange interaction is too weak and can handle the stability of ions only at relatively low temperatures (below 600°K). In some electron–ion models [16], the air is assumed to be excluded from the ball lightning internal space, and the electron–ion plasma is under atmospheric pressure. However, these types of models are stable only at ambient temperatures.

The physics of electron–positron systems is similar to that of the physics of electron–hole systems in semiconductors, particularly the interaction with light, life time, vacuum breakdown by the electric field, and other effects are described in a similar way. Therefore, it is worth looking if any quantum bound states of a large number of electrons and holes have been observed in semiconductors. The 1970 paper of Keldysh [17] was probably the first to mention the electron–hole drops and first order phase transition from electron–hole gas to liquid in semiconductors. Consequently, the free energy and phase diagrams for electron–hole liquid have been reported in a few papers [18, 19] including experimental data [20].

Each of the ground states of the electron–positron pair (positronium) and electron–hole pair (exciton) have its own Bohr radius a_0, which is the typical "size" of the charge pair. The luminescence from electron–hole drops [20] indicates the average distance between charges $d \sim a_0$, which is required for efficient electron–hole recombination. The electron–positron drops with similar properties $r_s = d/a_0 \sim 1$ were simulated recently [21, 22], they have very short life time too. For the purpose of the theory of ball lightning, we need to find a novel state with a long lifetime, and low density $r_s \gg 1$; such a state would be invisible in semiconductors, and it was never described in semiconductor literature to the best of my knowledge.

In the present study, I use a model similar to the Wigner lattice, see [23] Chap. 5.2, assuming that electrons and positrons oscillate near the sites of a cubic lattice. Thermal fluctuations of electrons and positrons balance the Coulomb attraction force. The advantage of this model is the electron and positron wave functions separated by a long distance, which suppresses the electron–positron annihilation rate. The solution shows that the condition $r_s \gg 1$ is fulfilled for the entire range of temperature (up to 14000°K) observed [8] in ball lightnings, and therefore the proposed model is stable against electron–positron annihilation.

The one component (e.g. electron) Wigner lattice describes electrons in solid and therefore charge density N/V and temperature T are independent parameters. Both Monte-Carlo simulations[24, 25] and lattice vibration model[26] predict melting of the Wigner lattice when

$$\Gamma \equiv \left(\frac{4\pi}{3}\frac{N}{V}\right)^{1/3}\frac{e^2}{T} \lesssim 160 \tag{1}$$

For the electron–positron cloud in air we need to find the equilibrium between the solid crystal state and the surrounding plasma evaporating from the crystal. Calculation gives $\Gamma \sim 10$, and therefore the equilibrium state is most probably "melted" Wigner crystal. In the same time analysis of lattice vibrations yet to be done for two components Wigner lattice.

I propose the evaporation of charge pairs as the main mechanism limiting the lifetime of the dual species Wigner crystal. The calculation follows the text book problem [27] §81. I found a reasonable evaporation rate for the entire range of temperature, and the lifetime in a range from a few hundred seconds at low temperature to a few seconds at high temperature. The instability towards the formation of high-density quantum liquid drops followed by electron–positron annihilation is also possible. The lifetime calculations for this mechanism are yet to be done.

In this study, I work on the quantum bound state of a system of N electrons and N positrons occupying lattice sites in volume V with distance $d = n^{-1/3}$, where $n = N/V$ is the density. I start the calculation with the equation of the wave function of the electron/positron $H\psi = E\psi$, with the spinless Hamiltonian, which is as follows

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \left[\sum_j \frac{e^2}{|\vec{r}-\vec{r}_j|} - \sum_k \frac{e^2}{|\vec{r}-\vec{r}_k|}\right] \tag{2}$$

where $\vec{r}_j$ runs over sites of same charge sign and $\vec{r}_k$ runs over sites of opposite charge sign.

The quantum mechanical problem can be solved using the variation method. Taking the trial wave function from the ground state of a harmonic oscillator

$$|\psi_0\rangle = \frac{\kappa^{3/2}}{\pi^{3/4}}e^{-\kappa^2 r^2/2},\ \langle\psi_0|\ \psi_0\rangle = 1 \tag{3}$$

for each term of the potential energy we obtain

$$\langle\psi|\ \frac{e^2}{|\vec{r}-\vec{r}_j|}\ |\psi\rangle = \frac{e^2}{|\vec{r}_j|}g(\kappa r_j),\ g(x) = \frac{4}{\sqrt{\pi}}\int_0^x x^2 dx e^{-x^2} \tag{4}$$

The final expression for the energy contains the well known kinetic energy of a harmonic oscillator plus summation of sites $\vec{r}_j$, which we replace by summation of neighboring shells as follows:

$$\langle\psi|\ H\ |\psi\rangle = \frac{\hbar^2}{2m}\frac{3}{2}\kappa^2 + \frac{e^2}{d}\sum_l \frac{u_l}{R_l}g(R_l\kappa d)\ . \tag{5}$$

Here, l is the index of the neighbor shell, u_l is the number of neighbors including the interaction sign, and R_l is the radius of the shell. Each neighbor shell l is a set of sites at the same distance from the origin $|\vec{r}_j| = R_l d$.

According to the variation method, Eq. (5) should be minimized with respect to κ. This leads to the equation

$$\kappa a_0 + \frac{8}{3\sqrt{\pi}}\sum_l u_l(R_l\kappa d)^2 e^{-(R_l\kappa d)^2} = 0, \tag{6}$$

where $a_0 = \hbar^2/me^2$ is the Bohr radius. In the range of interest

$$r_s \equiv d/a_0 \gg 1,\ \kappa a_0 \ll 1,\ \kappa d \gtrsim 1 \tag{7}$$

the solution is set by the exponent of the shell $l = 0$, and with a reasonable approximation

$$\kappa = \frac{r_d}{d},\ r_d = \frac{1}{R_0}\sqrt{\ln\frac{8}{3\sqrt{\pi}}|u_0|R_0 r_s} \approx 1.7\sqrt{\ln r_s} \tag{8}$$

The low-density limit $r_s \gg 1$ is the generic case for the Wigner lattice approach (please refer [23], Chap. 5).

The calculation of the higher energy levels becomes mathematically complicated. Fortunately, high-order harmonic wave functions with zero angular momentum give the minimum energy at the same κ, because it is set by the exponent in $g(x)$ in the potential energy term of Eq. (5). I will therefore approximate the energy of higher

Lattice Type	Lattice const	R_0	u_0	R_1	u_1	M
CsCl	bcc, 1	$\sqrt{3}/2$	-8	1	6	1.76
NaCl	fcc, $\sqrt[3]{4}$	$\sqrt[3]{4}/2$	-6	$\sqrt[3]{4}/\sqrt{2}$	12	1.74
ZnS	fcc, $\sqrt[3]{4}$	$\sqrt[3]{4}\sqrt{3}/4$	-4	$\sqrt[3]{4}/\sqrt{2}$	12	1.64

TABLE I. The Madelung constant M for few lattices; it can be formally defined as $M/R_0 = -\sum_l u_l/R_l$, where u_l is the signed number of nodes at a same distance R_l from the center, and l is the number of indexing neighbor shells. The sum of shells is divergent, however the sum of planes is convergent. [28] The lattice constant is chosen in such a way so as to have two nodes in a unit cell.

energy levels by the harmonic series with level separation per dimension as follows:

$$\Delta E = \frac{\hbar^2\kappa^2}{m} = \frac{\hbar^2 r_d^2}{md^2} \tag{9}$$

and offset $-U_0$, where U_0 is the potential energy in the limit $\kappa \to \infty$ in Eq. (5) as follows:

$$U_0 = -\frac{e^2}{d}\sum_l \frac{u_l}{R_l} = \frac{e^2}{d}\frac{M}{R_0} \tag{10}$$

The lattice sum in Eq. (10) converges asymptotically, Table I. gives the Madelung constant M for a few lattices.

The free energy F(N,V,T) for electrons (or positrons) is obtained by summing the oscillator states as follows:

$$F = -NT\ln\frac{e^{U_0/T}}{8\sinh^3(\Delta E/2T)} \approx -NU_0 + 3NT\ln\frac{\Delta E}{T} \tag{11}$$

where the high-temperature limit $T \gg \Delta E$ is justified. Indeed, a stable solution requires a balance between thermal fluctuations and the Coulomb attraction force

$$T \sim e^2/d \gg \Delta E \sim e^2 a_0/d^2 \ , \tag{12}$$

by virtue of Eq. (7), and for the same reason, the theory holds in a very broad range of temperatures.

$$e^2/a_0 \gg T > 0 \ . \tag{13}$$

The ground state at $T = 0$ should be excluded; the balance between quantum fluctuations and the Coulomb attraction force is possible only at $d \sim a_0$, violating the low-density assumption.

The substitution of Eq. (8) and $d = (N/V)^{-1/3}$ into Eq. (11) leads to

$$F = -\frac{e^2 M}{R_0}\frac{N^{4/3}}{V^{1/3}} + NT\ln\left[\left(\frac{\hbar^2 r_d^2}{mT}\right)^3 \frac{N^2}{V^2}\right] \ . \tag{14}$$

The obtained free energy Eqs. (11,14) follows general results for crystals at high temperature[27]§65, particularly it is characterized by a constant specific heat $C = 3N$. The electron (positron) partial pressure P and chemical potential μ are

$$P = -\frac{\partial F}{\partial V} = \frac{2NT}{V} - \frac{e^2 M}{3R_0}\left(\frac{N}{V}\right)^{4/3} \tag{15}$$

$$\mu = \frac{\partial F}{\partial N} = 2T + T\ln\left[\left(\frac{\hbar^2 r_d^2}{mT}\right)^3 \frac{N^2}{V^2}\right] - \frac{4e^2 M}{3R_0}\frac{N^{1/3}}{V^{1/3}} \tag{16}$$

The derivatives of r_d are neglected when computing P and μ. The chemical potential of Eq. (16) is analogous to the $\mu(n)$ curve for electron–hole systems in semiconductor[18]. Eq. (16) lacks the term $\mu \sim n^{2/3}$ originating from $1/r_s^2 \sim n^{2/3}$ in the ground state energy at zero temperature. This is because Eq. (16) was derived in the low–density approximation $r_s \gg 1$. The system with the free energy, represented by Eq. (11) has no liquid phase. The formation of the liquid phase requires a positive branch of $\mu(n)$ for $n \to \infty$.

Realistic ball lightning demonstrates little interaction with air and the ability to penetrate thin layers of solid matter. For this reason, air pressure can be neglected. The crystal part of the ball lightning must be surrounded by the gas of electrons and positrons evaporating from the crystal. This will be my model for further calculation of the crystal state properties and lifetime. I also assume that only the electron–positron pair in some excited states can leave the crystal while preserving the charge neutrality. The density of both the crystal n_c and the surrounding vapor of electron–positron pairs n_f is computed by matching the chemical potential, temperature, and partial pressure between the two phases.

The pressure of the vapor above the solid is typically small, and assuming $P = 0$, one finds the density of the electron–positron crystal as follows:

$$n_c = \left(\frac{6R_0}{M}\frac{T}{e^2}\right)^3 \ . \tag{17}$$

This point is on the raising branch of the $P(V)$ curve so it cannot exist alone for a long time. A positive pressure is required to drive the system into a stable equilibrium between the crystal and the surrounding vapor. The calculation of the exact coexistence curve $P(T)$ is not required for the purpose of this work, because the pressure of the surrounding vapor is not maintained and the electron–positron crystal evaporates.

The evaporated electron–positron pair should have a binding energy of T, and therefore a size of about d, and the moment of inertia is given by $I \sim md^2/2$. Then, the free energy for electron–positron pairs [27] §47 is as follows

$$F_f = -N_f T \ln\left[\frac{2.7V_f}{N_f}\left(\frac{mT}{\pi\hbar^2}\right)^{3/2}\frac{2IT}{\hbar^2}\right] \tag{18}$$

$$\mu_f = T\ln\left[\frac{N_f}{V_f}\left(\frac{\pi\hbar^2}{mT}\right)^{3/2}\frac{T}{E_0}\right] \tag{19}$$

$$\frac{2IT}{\hbar^2} \sim \frac{mT}{\hbar^2}\left(\frac{6R_0}{M}\frac{T}{e^2}\right)^{-2} \equiv \frac{E_0}{T} \tag{20}$$

Here the index f marks quantities that belong to evaporated electron–positron pairs, particularly $n_f = N_f/V_f$. I am not counting the spin degeneracy just for consistency with previous calculations. The matching of μ_f with twice μ from Eq. (16), and N/V obtained from Eq. (17) gives

$$\frac{n_f}{n_c} = \frac{r_d^{12}}{2.72^{12}\pi^{3/2}}\left(\frac{T}{E_0}\right)^{7/2} \equiv \left(\frac{T}{E_0'}\right)^{7/2} \tag{21}$$

where E_0' absorbs all constant factors. The evaporation rate per area [27] §81 is $\sqrt{T/m}\, n_f$, therefore

$$\dot{N} = -\frac{6N}{L}\sqrt{\frac{T}{m}}\left(\frac{T}{E_0'}\right)^{7/2}, \ \frac{N}{L^3T^3} = \text{const.} \tag{22}$$

where the surface area of a ball lightning is approximated as $6N/(n_c L)$, and L is the diameter connected to N and T by Eq. (21), and n_f is derived from Eq. (17).

The cooling of the ball lightning by evaporation must be accounted for in the proper life time calculation. The relevant exponent γ is as follows:

$$TN^{-\gamma} = \text{const}\ , \quad \gamma = \frac{(s_f/2 - s_c)}{c_v} = \frac{5}{4} \tag{23}$$

where s_f is the entropy per free electron–positron pair, and s_c and c_v are the entropy and the specific heat per electron in the crystal. The above equation means that a ball lightning loses its temperature (the cooling process) at almost teh same pace as the mass. Then, the solution to Eqs. (23,22) is

$$N = N_0\left(\frac{t}{t_0}+1\right)^{-0.17}, \quad \frac{0.17}{t_0} = \frac{6}{L_0}\sqrt{\frac{T_0}{m}}\left(\frac{T_0}{E_0'}\right)^{7/2} \tag{24}$$

where N_0, T_0, and L_0 are the initial values of the number of electrons N, temperature T, and ball lightning size L, respectively. Consider for example, a $L_0 = 100$cm ball lightning at a temperature of $T_0 = 5000°$C (see [9] interpretation of the data from [11]). The time of the drop of the temperature four times and the number of charges three times is 50s.

I conclude that the lifetime of the proposed electron–positron crystalline state with respect to evaporation is reasonable and comparable with the observed in nature. The lifetime with respect to electron–positron annihilation is yet to be calculated; I expect this process to be slow because of the small overlap between electron and positron wave functions in the crystal state. At the same time, the detection of 511 keV gamma rays originating from a ball lightning would confirm the present model.

The physics of the mixed Wigner lattice is very different from the physics of the repulsive Wigner lattice, where the distance is fixed and depends on the number of charges and the size of the system. For instance, we report a negative temperature expansion coefficient for the lattice state, because higher kinetic energy should be balanced by stronger attraction forces and closer distance between electrons and positrons.

The electron–positron plasma has been studied in detail for many years. Most studies have focused on high–density plasma in pulsars, and low–density plasma in various traps and plasma waves and excitation.[14, 15] Therefore I assume that the two species Wigner lattice is proposed for the first time in this work. Plasma traps typically have a focusing effect and promote the formation of positronium. The purpose of the present study is to find a long living state that prevents electron–positron pairing.

In brief, I calculated the free energy and the equation of state for neutral electron-positron clouds by using the Wigner lattice approach. This state of matter can be observed by registering 511 keV gamma rays coming from a ball lightning.

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