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Exchange of quantum fields by method of group embedding

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The exchange between N quantum fields is investigated by raising symmetry from $SO(1;3)$ to $SO(N;3N)$, rotating $4N$ dimensional space, and lowering the symmetry back to $SO(1;3)$. We track the exchange matrix for $SO(1;3)$ spinors (expected to be anticommuting) and products of even number of spinors (expected to be tensors and, therefore, commuting). For spinors, the theory turns out to be supersymmetric. Vector fields are commuting. The transformation takes a scalar field to a pseudoscalar and, therefore, commutation rules for scalars cannot be computed using this method.

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I. INTRODUCTION

The scientific interest in and debate over the foundation of the spin–statistics theorem[1] have continued since the theorems publication.[2] This theorem is required for understanding the stability of matter,[3] periodic filling of atomic orbitals, operation of lasers, and several other phenomena. The theorem states that an ensemble of several like particles will belong to the symmetrical class if the particles have integer spin, and it will belong to the antisymmetrical class if the particles have half-integer spin; a mixture of all symmetry classes is never realized in nature.[4]

The original proof of the theorem is based on the CPT-invariance of energy, charge, and Greens function. It makes use of the symmetry between particles and antiparticles.[5] However, one can argue that it should be possible to prove the spin–statistics theorem within elementary quantum mechanics.[6] The idea is to replace the particle exchange by rotation through additional degrees of freedom. The rotation of the wave function is a smooth operation that should provide unambiguous information about its sign.[7–13]

The geometrical arguments in favor of the spin–statistic theorem in most cases are formalizations of the belt trick.[14] There are a few difficulties in realizing this approach. The smooth exchange rotations of quantum fields suffer from singularities that occur in the construction of space translations.[7] The exchange is typically done through the $SU(2N)$ group; unfortunately, this produces unphysical states.[15]

The present paper makes use of a four-step procedure, which is outlined below, to connect the exchange of $SO(1;3)$ fields and rotations in $SO(N;3N)$ space. We start with the example of $N=2$ and then discuss a generalization to arbitrary N.

1. Embed the $SO(1;3)$ fields into one $SO(2;6)$ field.
2. Calculate the transformation of the $SO(2;6)$ field upon rotation-exchanging embedded subspaces.
3. Branch the $SO(2;6)$ field back to the $SO(1;3)$ fields.
4. The transformation of the $SO(2;6)$ field becomes an exchange of the $SO(1;3)$ fields.

The group theory structure of the above procedure is very different from that used in the past. We do not use any space translation groups and do not follow the belt trick. Instead, we go straight to higher dimensions to avoid difficulties with parasitic states and singularities.

Commutation rules can be computed between fields of the same symmetry such as between spinor and spinor fields or between vector and vector fields. For this reason, not all $SO(2;6)$ fields are suitable for this analysis. We will use $SO(2;6)$ bispinor, which branches to a product of $SO(1;3)$ bispinors. Moreover, the $SO(2;6)$ adjoint field branches to a product of vectors. Unfortunately, the $SO(2;6)$ scalar field branches to a product of an $SO(1;3)$ scalar and a pseudoscalar, and cannot be used for the calculation of commutation rules.

This work uses the unified approach, whereby all $SO(2;6)$ tensors are expressed through $SO(2;6)$ bispinors. The rotations of all tensors are then obtained from the transformation of bispinors. We keep all products of bispinor quantum fields normally-ordered to avoid ambiguity upon symmetry raising and lowering. The normal ordering is not shown explicitly through the calculations, but it will be always assumed.

When this work began, the author anticipated obtaining the spin–statistics theorem, that is, commutation of integer spin fields and anticommutation of half-integer fields. The calculations confirm the commutation of vector fields in accordance with the spin–statistics theorem. The result is surprising for spinor fields: spinor fields are found to be supersymmetric, with only left spinors anticommuting. This leads to the paradox mentioned in the Landau course [16, §26]. The products of an even number of spinors form vectors and tensors. Supersymmetric spinor fields would then lead to supersymmetric tensors.

This paradox is resolved by the following arguments. Vector fields are made by pairing anticommuting fields with anticommuting fields $\psi_L^\dagger\vec{\sigma}\psi_L$ and commuting fields with commuting fields $\psi_R^\dagger\vec{\sigma}\psi_R$, and therefore, they are bosons (no paradox). Scalar fields are made by pairing anticommuting fields with commuting $\psi_L^\dagger\psi_R$, and therefore they are candidates for anticommuting spin-zero fields. The exchange of scalar fields changes their symmetries to pseudoscalars, and therefore, they are neither commuting nor anticommuting.

The square of a scalar field is also a scalar field and therefore it must be commuting. The scalar product of two vector fields is a scalar field too and it must be commuting. The permutation of scalar field should therefore preserve their symmetry and this contradicts previous statement about the symmetry change.

This paradox is resolved by the following reasoning. The method of embedding of $SO(1;3)$ into $SO(N;3N)$ requires $SO(N;3N)$ to be the original world symmetry, spontaneously broken down to $SO(1;3)$. Here N is the total number of particles in the $SO(1;3)$ world and therefore the $SO(N;3N)$ world has only one particle. In this world there are no powers of $\psi(x)$; it possible to form scalar $s(x) = \psi^\dagger(x)\psi(x)$ but there is no way to form square of scalar field. Upon the symmetry lowering to the $SO(1;3)$ world the product of two scalars is allowed $s(x)s(x')$ but the square of a scalar is not allowed.

The exchange rotation between two subspaces of $SO(N;3N)$ is shown below in a few steps. We assume that one subspace has axes x_1^μ and the other subspace has axes x_2^μ, where $\mu_j = 0,1,2,3$. The first step is a $\pi/2$ rotation $\hat{R}_{\pi/2}$ in the four planes $x_1^\mu x_2^\mu$. This takes $x_1^\mu \to x_2^\mu$, $x_2^\mu \to -x_1^\mu$, etc. In other words, for an $SO(2;6)$ scalar one has

$$\hat{R}_{\pi/2}:\ s(x_1,x_2) \to s(x_2,-x_1) \tag{1}$$

Then we will need $\hat{I}_2$ which inverts the x_1^μ subspace such that $x_1^\mu \to -x_1^\mu$, etc:

$$\hat{I}_2:\ s(x_2,-x_1) \to s(x_2,x_1)\ . \tag{2}$$

Finally, the exchange rotation is defined as

$$\hat{E} = \hat{I}_2\hat{R}_{\pi/2}:\ s(x_1,x_2) \to s(x_2,x_1)\ , \quad \hat{E}^2 = 1 \tag{3}$$

The exchange of subspaces is not a smooth transformation, because it includes inversion in one of the subspaces. However, all the elements are unambiguously defined, and therefore, the exchange matrix is expected to be well defined.

Section II defines bispinor fields for $SO(1;3)$ and $SO(2;6)$ and presents the results of the exchange rotation for bispinors. All γ-matrix-related algebra, rotation matrices, and certain other calculations are included in the appendices. The following sections (Sec. III and IV) repeat the procedure of Sec. II for integer spin fields. The metric tensor is not included because it requires four bispinors for construction, resulting in overly heavy calculations. The N-dimensional generalization of the theory is discussed shortly in Sec. V, after which the work is summarized in Sec. VI.

II. THE EXCHANGE ROTATION OF BISPINORS

We start with half-integer spin fields because their embedding in a higher-dimensional group is relatively easy. For the $SO(1;3)$ and $SO(2;6)$ bispinors and their conjugates, we obtain

$$\begin{aligned}
&[1,0] = \psi_L(x),\ \psi_R^\dagger(x) \\
&[0,1] = \psi_R(x),\ \psi_L^\dagger(x) \\
&[0,0,1,0] = \psi_L(x,x'),\ \psi_L^\dagger(x,x') \\
&[0,0,0,1] = \psi_R(x,x'),\ \psi_R^\dagger(x,x')\ ,
\end{aligned} \tag{4}$$

where L indicates left spinors and R indicates right spinors. The highest Dynkin weights in square brackets denote the $D(2)$ and $D(4)$ left and right spinors.[17] . We will not use $\bar{\psi}$ notations in this work.

The embedding (branching) rules are

$$\begin{aligned}
\psi_L(x,x') &\leftrightarrow \psi_L(x)\otimes\psi_R(x') \oplus \psi_R(x)\otimes\psi_L(x') \\
\psi_R(x,x') &\leftrightarrow \psi_L(x)\otimes\psi_L(x') \oplus \psi_R(x)\otimes\psi_R(x')\ ,
\end{aligned} \tag{5}$$

and similar rules apply for conjugated fields. Bispinors for both groups are defined as

$$\psi(x) = \psi_L(x) \oplus \psi_R(x) \tag{6}$$

$$\psi(x,x') = \psi_L(x,x') \oplus \psi_R(x,x')\ . \tag{7}$$

The exchange rotation transforms the $SO(2;6)$ bispinor $\psi(x,x')$ as

$$\psi(x,x') \to \hat{E}\psi(x',x)\ . \tag{8}$$

We postulate in this work that the matrix $\hat{E}$ is preserved when the symmetry lowered from $SO(2;6)$ to $SO(1;3)$. It allows the calculation of the exchange of $SO(1;3)$ bispinors

$$\psi(x)\psi(x') \to \hat{E}\psi(x')\psi(x)\ . \tag{9}$$

The explicit calculation in A gives the action of the exchange rotation

$$\begin{matrix}
\psi_{LL}(x,x') & & -\psi_{LL}(x',x) \\
\psi_{LR}(x,x') & \to & \psi_{RL}(x',x) \\
\psi_{RL}(x,x') & & \psi_{LR}(x',x) \\
\psi_{RR}(x,x') & & \psi_{RR}(x',x)
\end{matrix} \tag{10}$$

and after lowering the symmetry we obtain

$$\begin{matrix}
\psi_L(x)\psi_L(x') & & -\psi_L(x')\psi_L(x) \\
\psi_L(x)\psi_R(x') & \to & \psi_R(x')\psi_L(x) \\
\psi_R(x)\psi_L(x') & & \psi_L(x')\psi_R(x) \\
\psi_R(x)\psi_R(x') & & \psi_R(x')\psi_R(x)
\end{matrix} \tag{11}$$

meaning that left spinors anticommute, and all other commute. The same rules apply for conjugated fields because the exchange matrix is real.

III. CONSTRUCTION OF TENSORS FROM SPINORS

For tensor fields, the generic embedding rules are slightly more involved. It is convenient to calculate all rotations by making use of γ-matrices; therefore, we will express the tensors in terms of bispinors. We will distinguish tensors and pseudotensors by making use of the γ^0 matrix in $SO(1;3)$ and γ_1^0 in $SO(2;6)$. If the transformations $\psi \to \gamma^0\psi$ or $\psi \to \gamma_1^0\psi$ change the sign of a tensor we will call it a pseudotensor.

It should be noted that Appendix A defines γ-matrices for all $SO(N;3N)$. We omit N in the γ-matrix notation, assuming that the dimension of the γ-matrices matches that of the spinor fields ψ in every equation. In particular, the γ-matrices in Eq. (12) are 4×4 and in Eq. (16) they are 16×16, acting on both subspaces of $SO(2;6)$. Scalar $SO(1;3)$ fields are

$$[0,0] = s(x) = \psi^\dagger(x)\gamma^{\text{ZERO}}\psi(x)\ ,$$
$$[0,0]_p = s_p(x) = \psi^\dagger(x)\gamma^{\text{ZERO}}\gamma^{\text{FIVE}}\psi(x)\ , \qquad (12)$$

where the subscript p means pseudoscalar, and will be used to indicate pseudovectors as well. The vector fields are given by

$$[1,1] = \phi^\mu(x) = \psi^\dagger(x)\gamma^{\text{ZERO}}\gamma^\mu\psi(x)\ , \qquad (13)$$
$$[1,1]_p = \phi_p^\mu(x) = \psi^\dagger(x)\gamma^{\text{ZERO}}\gamma^{\text{FIVE}}\gamma^\mu\psi(x)\ .$$

The adjoint representation is the last tensor field constructed from just two spinors

$$[2,0]\oplus[0,2] = \sigma^{\mu\nu}(x) = \psi^\dagger(x)\gamma^{\text{ZERO}}\gamma^\mu\gamma^\nu\psi(x)\ ,$$
$$\tilde\sigma^{\mu\nu}(x) = \psi^\dagger(x)\gamma^{\text{ZERO}}\gamma^{\text{FIVE}}\gamma^\mu\gamma^\nu\psi(x) \qquad (14)$$
$$= -(i/2)\epsilon^{\mu\nu\mu'\nu'}\sigma_{\mu'\nu'}(x)\ ,$$

where $\epsilon^{\mu\nu\mu'\nu'}$ is given by Eq. (B6). Because the field $\sigma^{\mu\nu}(x)$ is odd with respect to permutations of indexes μ,ν, the field $\tilde\sigma^{\mu\nu}(x)$ is the same field with different indexing of components.

Overall, we decompose the product of bispinors to a sum of five tensor fields $4\cdot 4 = 1 + 1_p + 4 + 4_p + 6$. The metric tensor (e.g. gravitational field) $[2,2] = h^{\mu\nu}(x)$ appears in the decomposition of products of four bispinors.

$$[2,2] = [2,0]\otimes[0,2] \qquad (15)$$
$$h(x) = (\sigma_1^{\mu\mu'}(x) - \tilde\sigma_1^{\mu\mu'}(x))(\sigma_2^{\nu\nu'}(x) + \tilde\sigma_2^{\nu\nu'}(x))$$

However decomposition of the field into product of four spinors is forbidden because the high symmetry state can have not more than one particle.

The index μ for $SO(1;3)$ tensors ranges from 0 to 3; we will use same index for $SO(2;6)$, together with an extra index j in the range from 1 to 2; it will count the $SO(2;6)$ subspaces. The scalar $SO(2;6)$ field is

$$[0,0,0,0] = s(x,x') = \psi^\dagger(x,x')\gamma^{\text{ZERO}}\psi(x,x'), \qquad (16)$$

because γ_1^0 commutes with γ^{ZERO} for all N. The $SO(2;6)$ pseudoscalar is therefore

$$[0,0,0,0]_p = s_p(x,x') = \psi^\dagger(x,x')\gamma^{\text{ZERO}}\gamma^{\text{FIVE}}\psi(x,x'). \qquad (17)$$

The vector fields are

$$[1,0,0,0] = \phi_j^\mu(x,x') \ = \ \psi^\dagger(x,x')\gamma^{\text{ZERO}}\gamma_j^\mu\psi(x,x') \qquad (18)$$
$$[1,0,0,0]_p = \phi_{pj}^\mu(x,x') \ = \ \psi^\dagger(x,x')\gamma^{\text{ZERO}}\gamma^{\text{FIVE}}\gamma_j^\mu\psi(x,x')$$

The adjoint representations (antisymmetric tensor) are

$$[0,1,0,0] \ = \ \sigma_{jj'}^{\mu\mu'}(x,x')\psi^\dagger(x,x')\gamma^{\text{ZERO}}\gamma_j^\mu\gamma_{j'}^{\mu'}\psi(x,x'),$$
$$[0,1,0,0]_p \ = \ \sigma_{pjj'}^{\mu\mu'}(x,x') \qquad (19)$$
$$= \ \psi^\dagger(x,x')\gamma^{\text{ZERO}}\gamma^{\text{FIVE}}\gamma_j^\mu\gamma_{j'}^{\mu'}\psi(x,x'),$$

where $\mu j \neq \mu' j'$. The products of the three γ-matrices give

$$[0,0,1,1] \ = \ v_{jj'k}^{\mu\mu'\nu}(x,x')$$
$$= \ \psi^\dagger(x,x')\gamma^{\text{ZERO}}\gamma_j^\mu\gamma_{j'}^{\mu'}\gamma_k^\nu\psi(x,x'),$$
$$[0,0,1,1]_p \ = \ v_{pjj'k}^{\mu\mu'\nu}(x,x') \qquad (20)$$
$$= \ \psi^\dagger(x,x')\gamma^{\text{ZERO}}\gamma^{\text{FIVE}}\gamma_j^\mu\gamma_{j'}^{\mu'}\gamma_k^\nu\psi(x,x').$$

where $\mu j \neq \mu' j' \neq \nu k$. The last field made from two representations is

$$[0,0,2,0]\oplus[0,0,0,2] = u_{jj'kk'}^{\mu\mu'\nu\nu'}(x,x')$$
$$= \psi^\dagger(x,x')\gamma^{\text{ZERO}}\gamma_j^\mu\gamma_{j'}^{\mu'}\gamma_k^\nu\gamma_{k'}^{\nu'}\psi(x,x'), \qquad (21)$$

Overall, we decompose the product of the $SO(2;6)$ bispinors to a sum of nine tensor fields $16\cdot 16 = 1 + 1_p + 8 + 8_p + 28 + 28_p + 56 + 56_p + 70$, where the subscript p indicates a pseudotensor.

IV. THE EXCHANGE ROTATION OF TENSORS

The transformation of the $SO(2;6)$ tensor fields is computed by the action of the operator $\hat E$ on the function $\psi(x,x')$. The operator $\hat E$ should be substituted into Eqs. (17–21), followed by the rotation of the γ-matrices by $\hat E$.

Before studying the exchange rotations of the $SO(2;6)$ tensors, let us summarize all counts of degrees of freedom upon lowering the symmetry from $SO(2;6)$ to $SO(1;3)$. The explicit calculation is shown in Appendix B:

$$1 \to 1_p\cdot 1\ , \quad 1_p \to 1\cdot 1_p\ , \qquad (22a)$$
$$8 \to 4_p\cdot 1_p + 1_p\cdot 4\ , \quad 8_p \to 4\cdot 1 + 1\cdot 4_p\ , \qquad (22b)$$
$$28 \to 6\cdot 1 + 4_p\cdot 4_p + 1_p\cdot 6\ , \qquad (22c)$$
$$28_p \to 6\cdot 1_p + 4\cdot 4 + 1\cdot 6\ , \qquad (22d)$$
$$56 \to 4\cdot 1_p + 6\cdot 4 + 4_p\cdot 6 + 1_p\cdot 4_p\ ,$$
$$56_p \to 4_p\cdot 1 + 6\cdot 4_p + 4\cdot 6 + 1\cdot 4\ , \qquad (22e)$$
$$70 \to 1\cdot 1 + 4\cdot 4_p + 6\cdot 6 + 4_p\cdot 4 + 1_p\cdot 1_p\ . \qquad (22f)$$

The exchange between fields of the same symmetry is possible by rotation of representations with 28 and 70 degrees of freedom.

Let us calculate the transformation of the 28-component adjoint representation, which contains a product of vectors. We will see symmetry change of some representations upon exchange, and for comparison we puit first the transformation of the 1-component scalar field:

$$s(x,x') \;\to\; \psi(x',x)\hat{E}\gamma^{\text{ZERO}}\hat{E}\psi(x',x). \tag{23}$$

$$\sigma_{12}^{\mu\mu'}(x,x') \;\to\; \psi(x',x)\hat{E}\gamma^{\text{ZERO}}\gamma_1^{\mu}\gamma_2^{\mu}\hat{E}\psi(x',x), \tag{24}$$

Taking into account that in $SO(2;6)$

$$\begin{aligned} &\gamma_1^{\mu} \to \hat{E}\gamma_1^{\mu}\hat{E} = \gamma_2^{\mu}\ , \quad \gamma_2^{\mu} \to \hat{E}\gamma_2^{\mu}\hat{E} = \gamma_1^{\mu} \\ &\gamma^{\text{FIVE}} \to \hat{E}\gamma^{\text{FIVE}}\hat{E} = \gamma^{\text{FIVE}} \\ &\gamma^{\text{ZERO}} \to \hat{E}\gamma^{\text{ZERO}}\hat{E} = -\gamma^{\text{ZERO}} \end{aligned} \tag{25}$$

we arrive at the exchange rotation matrix for $SO(2;6)$ adjoint and scalar

$$\begin{matrix} s(x,x') \\ \sigma_{11}^{\mu\mu'}(x,x') \\ \sigma_{22}^{\mu\mu'}(x,x') \\ \sigma_{12}^{\mu\mu'}(x,x') \end{matrix} \quad \to \quad \begin{matrix} -s(x',x) \\ -\sigma_{22}^{\mu\mu'}(x',x) \\ -\sigma_{11}^{\mu\mu'}(x',x) \\ -\sigma_{21}^{\mu\mu'}(x',x) \end{matrix}\ , \tag{26}$$

and after lowering the symmetry we obtain

$$\begin{matrix} s_p(x)s(x') \\ \tilde{\sigma}_{\mu\mu'}(x)s(x') \\ s_p(x)\sigma^{\mu\mu'}(x') \\ \\ \phi_p^{\mu}(x)\phi_p^{\mu'}(x') \end{matrix} \quad \to \quad \begin{matrix} -s_p(x')s(x) \\ -s_p(x')\sigma^{\mu\mu'}(x) \\ -\tilde{\sigma}_{\mu\mu'}(x')s(x) \\ \\ \phi_p^{\mu'}(x')\phi_p^{\mu}(x) \end{matrix} \tag{27}$$

From this, we learn that pseudovector fields $\phi_p^{\mu}(x)$ commute as expected. A similar result is obtained from the rotation of the pseudoadjoint field

$$\sigma_{p12}^{\mu\mu'}(x,x') \quad \to \quad -\sigma_{p21}^{\mu\mu'}(x',x)\ , \tag{28}$$

$$\phi^{\mu}(x)\phi^{\mu'}(x') \quad \to \quad \phi^{\mu'}(x')\phi^{\mu}(x) \tag{29}$$

which confirms the commutation of vectors $\phi^{\mu}(x)$.

The representation with 70 degrees of freedom is transformed as

$$\begin{matrix} u_{1111}^{\mu\mu'\nu\nu'}(x,x') \\ u_{1112}^{\mu\mu'\nu\nu'}(x,x') \\ u_{1122}^{\mu\mu'\nu\nu'}(x,x') \\ u_{1222}^{\mu\mu'\nu\nu'}(x,x') \\ u_{2222}^{\mu\mu'\nu\nu'}(x,x') \end{matrix} \quad \to \quad \begin{matrix} -u_{2222}^{\mu\mu'\nu\nu'}(x',x) \\ -u_{2221}^{\mu\mu'\nu\nu'}(x',x) \\ -u_{2211}^{\mu\mu'\nu\nu'}(x',x) \\ -u_{2111}^{\mu\mu'\nu\nu'}(x',x) \\ -u_{1111}^{\mu\mu'\nu\nu'}(x',x) \end{matrix}\ . \tag{30}$$

The choice of indexes $\mu\nu\mu'\nu'$ is limited; they should be different if they belong to the same subspace. Upon symmetry lowering, we obtain the following exchange rules

$$\begin{matrix} s_p(x)s_p(x') \\ \phi_{p\kappa}(x)\phi^{\nu'}(x') \\ \sigma^{\mu\mu'}(x)\tilde{\sigma}^{\nu\nu'}(x') \\ \phi^{\mu}(x)\phi_{p\kappa'}(x') \\ s(x)s(x') \end{matrix} \quad \to \quad \begin{matrix} -s(x')s(x) \\ \phi^{\nu'}(x')\phi_{p\kappa}(x) \\ -\sigma^{\nu\nu'}(x')\tilde{\sigma}^{\mu\mu'}(x) \\ \phi_{p\kappa'}(x')\phi^{\mu}(x) \\ -s_p(x')s_p(x) \end{matrix}\ , \tag{31}$$

where $\epsilon^{\mu\mu'\nu\kappa} = 1$ and $\epsilon^{\kappa'\mu'\nu\nu'} = 1$ define a relationship between the indexes in Eq. (31) and Eq. (30).

We see from Eqs. (27,31) that a scalar field becomes a pseudoscalar field upon the exchange, a product of scalars becomes a product of pseudoscalars, and therefore they neither commute nor anticommute. A similar problem occurs for the exchange of tensors $\sigma^{\mu\mu'}$; indexes of this field get shuffled $\sigma^{\mu\mu'} \to \tilde{\sigma}^{\mu\mu'}$, which means that the components of the tensor $\sigma^{\mu\mu'}$ neither commute nor anticommute.

V. GENERALIZATION TO N FIELDS

The exchange of any two out of N fields requires that the symmetry be raised to $SO(N;3N)$ and lowered back to $SO(1;3)$. This is relatively easy for bispinors (see Appendix A). The exchange of vectors between i and j subspaces can be done in few ways depending on the choice of $SO(N;3N)$ field. Fields of the type $w = \psi^{\dagger N}\gamma^{\text{ZERO}}(\prod_{k\mu}\gamma_k^{\mu})\psi^N$ will contain a product of two vectors $\phi_i^{\mu}(x_i)$ and $\phi_j^{\mu'}(x_j)$ after symmetry lowering, if the index k run one or three times over i and one or three times over j.

Upon the exchange rotation γ^{ZERO} in w will change sign because of the permutation of γ_i^0 with γ_j^0. Another sign flip will occur upon symmetry lowering because of the permutation of one or three matrices γ_i^{μ} with one or three matrices $\gamma_j^{\mu'}$. The overall calculation is a straightforward generalization of Eqs. (28,29), which leads to the commutation of vectors as expected from the spin–statistics theorem.

VI. DISCUSSION AND CONCLUSIONS

In summary, we were able to calculate the exchange of $SO(1;3)$ spinor fields and vector fields by making use of rotations through higher dimensions. While vector fields are shown to be commuting as expected from the spin–statistics theorem, spinor fields are turning out to be supersymmetric. These results are consistent, because vector fields can be formed by products of spinors with the same commutation rules.

The payment for supersymmetry comes with the commutation rules of tensors with an even number of indexes. The commutation of adjoint fields shuffles their components, and therefore, they neither commute nor anticommute. The commutation of scalar fields changes them to pseudoscalar fields, and therefore, they neither commute nor anticommute.

Exchange of $SO(4)$ fields by raising symmetry to $SO(8)$ gives exactly the same result as that for $SO(1;3)$: the spinor fields are supersymmetric with only left spinors anticommuting. However, an $SO(8)$ scalar is decomposed to produce two $SO(4)$ scalars that commute upon exchange rotation. It also takes $SO(4)$ vectors to pseudovectors, and therefore $SO(4)$ vector fields are neither commuting nor anticommuting.

The supersymmetry of spinors is not a consequence of Lorenz invariance, because it holds for both $SO(1;3)$ and $SO(4)$. It originates from the anticommutation of γ-matrices between subspaces. It is possible to construct an exchange operator within the $SO(1;3)$ group by utilizing commuting spin operators. (The spin operator of one spinor field commutes with the spin operator of the other spinor field.) The result is known as the Heisenberg Hamiltonian, and it describes anticommuting fields.

Previous works related to the connection between spin and statistics has focused on the topological foundations of the spin–statistics theorem.[7] These approaches suffer from the appearance of unphysical states and parasitic statistics.[15] The method of exchange rotation through $SO(N;3N)$ is free of unphysical states; $SO(N;3N)$ bispinors branch precisely to direct products of $SO(1;3)$ bispinors. At the same time, we abandon the concept of a non-relativistic proof of the spin–statistics theorem.

For future research we leave the foundation of the method of embedding of $SO(1;3)$ into $SO(N;3N)$; this requires $SO(N;3N)$ to be the original world symmetry, spontaneously broken down to $SO(1;3)$. The idea is to drive the spontaneous symmetry break by energy associated with the $SO(N;3N)$ metric field. The metric field becomes the order parameter and can drive the symmetry lowering. The change in the metric should describe how the $4N$-dimensional light cone becomes a composition of N four-dimensional light cones. Then one should integrate out fields that belong to $N-1$ instances of $SO(1;3)$. The resulting $SO(1;3)$ Lagrangian will have particle permutation laws, as computed in the present work. It is also possible that the symmetry break of $SO(N;3N)$ will give birth to more complicated symmetry groups and therefore generate internal symmetries of particles.

Present theory admits only five tensor fields: scalar, pseudoscalar, vector, pseudovector and adjoint. Only these five fields can be obtained upon the symmetry lowering of the $SO(N;3N)$ single particle spinor field. Other tensors, e.g. the gravity field, require four spinors to form and cannot be obtained by symmetry lowering of the $SO(N;3N)$ single particle field.

The deviation from the spin–statistics theorem can be interpreted as stating that matter at high energy is made from anticommuting left fields and antimatter from the commuting right field. Then, the symmetry transformation between matter and antimatter would anticommute with exchange. The lack of an exclusion principle for antimatter would explain the matter–antimatter asymmetry of the universe through the instability of antimatter.[3]

APPENDIX A: EXCHANGE OF 4-DIMENSIONAL SUBSPACES INSIDE $4N$-DIMENSIONS

The theory starts with ψ^N and $\psi^{\dagger N}$ being field operators in $SO(N;3N)$ space; they transform as 2^{2N} component bispinors of the D_{2N} Lie group

$$\psi^N = \psi^N_L \oplus \psi^N_R \sim \begin{pmatrix} [0,0,\ldots,1,0] \\ [0,0,\ldots,0,1] \end{pmatrix}, \qquad \text{(A1)}$$

and the same structure holds for $\psi^{\dagger N}$. Here the highest Dynkin weights in square brackets denote the $D(2N)$ left (also marked by the subscript "L") and right spinors (also marked by the subscript "R").[17]

The transformations of ψ^N upon space rotation are given in terms of the set of $4N$ anticommuting γ-matrices. The dimension of γ-matrices is $2^{2N} \times 2^{2N}$. In all equations in this work we assume that the dimension of the γ-matrices matches the dimension of the nearby ψ^N field. This allows to omit N in the notation of γ-matrices.

Here, $4N$ axes are counted as groups of four (subspaces). The index j counts groups, and the index μ counts axes within each group. Within each group, we use the space metrics $(1,-1,-1,-1)$. Thus, the explicit way of writing the γ-matrices is

$$\gamma^0_j = \underbrace{1\otimes\ldots\otimes 1}_{2(j-1)}\otimes\sigma_1\otimes\underbrace{\sigma_3\ldots\otimes\sigma_3}_{2(N-j)+1} \qquad \text{(A2a)}$$

$$\gamma^3_j = i\,\underbrace{1\otimes\ldots\otimes 1}_{2(j-1)}\otimes\sigma_2\otimes\underbrace{\sigma_3\ldots\otimes\sigma_3}_{2(N-j)+1} \qquad \text{(A2b)}$$

$$\gamma^2_j = i\,\underbrace{1\otimes\ldots\otimes 1}_{2(j-1)+1}\otimes\sigma_1\otimes\underbrace{\sigma_3\ldots\otimes\sigma_3}_{2(N-j)} \qquad \text{(A2c)}$$

$$\gamma^1_j = -i\,\underbrace{1\otimes\ldots\otimes 1}_{2(j-1)+1}\otimes\sigma_2\otimes\underbrace{\sigma_3\ldots\otimes\sigma_3}_{2(N-j)} \qquad \text{(A2d)}$$

and a few auxiliary matrices are

$$\gamma^{\text{FIVE}}_j = -i\gamma^0_j\gamma^1_j\gamma^2_j\gamma^3_j \qquad \text{(A2e)}$$

$$\gamma^{\text{FIVE}} = \prod_j \gamma^{\text{FIVE}}_j = \underbrace{\sigma_3\otimes\ldots\otimes\sigma_3}_{2N} \qquad \text{(A2f)}$$

$$\gamma^{\text{ZERO}} = \gamma^0_N\cdots\gamma^0_1(\gamma^{\text{FIVE}})^{N-1} \qquad \text{(A2g)}$$
$$\gamma^{\text{ZERO}}U^{-1} = U^\dagger\gamma^{\text{ZERO}}\ .$$

where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. As in the case of $N=1$, the matrices $\gamma^0_j, \gamma^1_j, \gamma^3_j$ are real, and the matrices γ^2_j are imaginary. The space rotation matrix U is introduced below. For $N=1$, we will omit the subscript indexes.

The space rotation in the plane $(j\mu, j'\mu')$ by angle θ is equivalent to the linear transformation of the $4N$ coordinates $x \to Rx$ and 2^{2N} components of the wave function

$$\psi^N(x) \;\to\; U\psi^N(R^{-1}x)\ , \quad U = e^{(\theta/2)\gamma^\mu_j\gamma^{\mu'}_{j'}} \qquad \text{(A3a)}$$
$$\psi^{\dagger N}(x) \;\to\; \psi^{\dagger N}(R^{-1}x)U^\dagger\ . \qquad \text{(A3b)}$$

The action of rotation operator R can be verified by

$$U\gamma_j^\mu x_{j\mu} U^{-1} = \gamma_j^\mu (Rx)_{j\mu} \ . \tag{A3c}$$

Finally, with the definition Eq. (A2g) one obtains the $4N$ "Lorenz"-invariant scalar field

$$s(x_1 \ldots x_N) = \psi^{\dagger N}(x_1 \ldots x_N)\gamma^{\text{ZERO}}\psi^N(x_1 \ldots x_N) \tag{A3d}$$

In other words $s(x_1 \ldots x_N)$ is invariant upon rotations in $4N$-dimensional space.

There is a transformation that we call the $i \leftrightarrow j$ subspace exchange

$$\forall\mu \; E_{ij}\gamma_i^\mu E_{ij} = \gamma_j^\mu \ , \quad E_{ij}\gamma_j^\mu E_{ij} = \gamma_i^\mu \tag{A4}$$

and our purpose is to calculate the field transformation by this subspace exchange

$$\psi^{N\prime}(\ldots x_i \ldots x_j \ldots) = E_{ij}\psi^N(\ldots x_j \ldots x_i \ldots) \ . \tag{A5}$$

It can be derived as sequence of $\pi/2$ rotations and four-inversion

$$E_{ij} = \frac{1}{4}(1+\gamma_i^0\gamma_j^0)(1-\gamma_i^1\gamma_j^1)(1-\gamma_i^2\gamma_j^2)(1-\gamma_i^3\gamma_j^3)\gamma_i^{\text{FIVE}} \tag{A6}$$

where

$$e^{\pm(\pi/4)\gamma_i^\mu\gamma_j^\mu} = \frac{1 \pm \gamma_i^\mu\gamma_j^\mu}{\sqrt{2}} \ , \tag{A7}$$

and the sign of rotation in plane $(i0, j0)$ is opposite to that of the rotations in planes $(i1, j1) \ldots (i3, j3)$. This can be verified by computing E_{ij} for $SO(4N)$.

The symbolic calculation was made by utilizing the cloud version of SAGE. The result in the form of Eq. (A5) is

$$\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\begin{matrix}
\psi_{\ldots\uparrow R\ldots\uparrow R\ldots}\\
\psi_{\ldots\uparrow R\ldots\downarrow L\ldots}\\
\psi_{\ldots\uparrow R\ldots\uparrow L\ldots}\\
\psi_{\ldots\uparrow R\ldots\downarrow R\ldots}\\
\psi_{\ldots\downarrow L\ldots\uparrow R\ldots}\\
\psi_{\ldots\downarrow L\ldots\downarrow L\ldots}\\
\psi_{\ldots\downarrow L\ldots\uparrow L\ldots}\\
\psi_{\ldots\downarrow L\ldots\downarrow R\ldots}\\
\psi_{\ldots\uparrow L\ldots\uparrow R\ldots}\\
\psi_{\ldots\uparrow L\ldots\downarrow L\ldots}\\
\psi_{\ldots\uparrow L\ldots\uparrow L\ldots}\\
\psi_{\ldots\uparrow L\ldots\downarrow R\ldots}\\
\psi_{\ldots\downarrow R\ldots\uparrow R\ldots}\\
\psi_{\ldots\downarrow R\ldots\downarrow L\ldots}\\
\psi_{\ldots\downarrow R\ldots\uparrow L\ldots}\\
\psi_{\ldots\downarrow R\ldots\downarrow R\ldots}
\end{matrix}
=
\begin{matrix}
\psi_{\ldots\uparrow R\ldots\uparrow R\ldots}\\
\psi_{\ldots\downarrow L\ldots\uparrow R\ldots}\\
\psi_{\ldots\uparrow L\ldots\uparrow R\ldots}\\
\psi_{\ldots\downarrow R\ldots\uparrow R\ldots}\\
\psi_{\ldots\uparrow R\ldots\downarrow L\ldots}\\
-\psi_{\ldots\downarrow L\ldots\downarrow L\ldots}\\
-\psi_{\ldots\uparrow L\ldots\downarrow L\ldots}\\
\psi_{\ldots\downarrow R\ldots\downarrow L\ldots}\\
\psi_{\ldots\uparrow R\ldots\uparrow L\ldots}\\
-\psi_{\ldots\downarrow L\ldots\uparrow L\ldots}\\
-\psi_{\ldots\uparrow L\ldots\uparrow L\ldots}\\
\psi_{\ldots\downarrow R\ldots\uparrow L\ldots}\\
\psi_{\ldots\uparrow R\ldots\downarrow R\ldots}\\
\psi_{\ldots\downarrow L\ldots\downarrow R\ldots}\\
\psi_{\ldots\uparrow L\ldots\downarrow R\ldots}\\
\psi_{\ldots\downarrow R\ldots\downarrow R\ldots}
\end{matrix} \tag{A8}$$

where we show only relevant channels. The rest of the matrix is diagonal. One can verify Eq. (A4) explicitly or by making use of SAGE. As expected, the exchange matrix E_{ij} does not contain only -1. After symmetry lowering of ψ^N to products of ψ, certain components will commute while others will anticommute.

The next task is to index the components of the $SO(N;3N)$ fields ψ_L^N, ψ_R^N according to the representations of the embedded group $SO(1;3) \oplus \cdots \oplus SO(1;3)$. We will prove that the exchange operator E has negative eigenvalues for the exchange of the left fields only (as shown above). The spinor ψ^N has 2^{2N} components; we will mark them by N handedness indexes L or R and by N spin indexes $\uparrow$ or $\downarrow$.

The common definition of the left and right spinors is derived from their sign upon the $4N$ inversion; in spinor representation the γ^{FIVE} is the operator of the $4N$ inversion, as $\forall j\mu: \quad \gamma^{\text{FIVE}}\gamma_j^\mu\gamma^{\text{FIVE}} = -\gamma_j^\mu$. Then the components of the spinor Eq.(A1) are selected by

$$\psi_L^N = \frac{1-\gamma_j^{\text{FIVE}}}{2}\psi^N \ , \quad \psi_R^N = \frac{1+\gamma_j^{\text{FIVE}}}{2}\psi^N \ , \tag{A9}$$

and ψ_L^N will change its sign upon $4N$ inversion, while ψ_R^N will preserve tits sign.

For every N, we can extract the index (L or R), which belongs to the j-th subspace by making use of γ_j^{FIVE} – the operator of four-inversion of the j-th subspace

$$\psi_{\ldots L\ldots} = \frac{1-\gamma_j^{\text{FIVE}}}{2}\psi^N \tag{A10a}$$

$$\psi_{\ldots R\ldots} = \frac{1+\gamma_j^{\text{FIVE}}}{2}\psi^N \ . \tag{A10b}$$

For example, the component of ψ branched to the product of only the left fields is

$$\psi_{L\ldots L} = \left(\prod_{j=1}^{N} \frac{1-\gamma_j^{\text{FIVE}}}{2} \right) \psi^N \ . \tag{A10c}$$

The N spin indexes can be extracted according to the sign of the generator of rotation in the $(j1, j2)$ plane:

$$\psi_{\ldots\uparrow\ldots} = \frac{1+i\gamma_j^1\gamma_j^2}{2}\psi^N \tag{A10d}$$

$$\psi_{\ldots\downarrow\ldots} = \frac{1-i\gamma_j^1\gamma_j^2}{2}\psi^N \ . \tag{A10e}$$

This labeling scheme allows us to write explicitly the action of the exchange operator in Eq. (A6). Exactly the same equation holds for $\psi^{\dagger N}$.

APPENDIX B: BRANCHING RULES FOR $SO(2;6)$ TENSORS UPON SYMMETRY LOWERING TO $SO(1;3)$

We start the calculations with the decomposition of the bispinors $\psi^\dagger(x,x') \to \psi^\dagger(x) \otimes \psi^\dagger(x')$, which will match the decomposition of the γ-matrices. In this work and particularly in this section, we will assume that all products (explicit and implicit) of ψ operators are normally-ordered. This allows us to avoid commutations between $\psi^\dagger$ and ψ.

The decomposition of the products of γ_j^μ matrices is required to calculate the branching rules of the $SO(2;6)$ tensors, and they are

$$\gamma^{\text{ZERO}}_{\text{SO}(2;6)} = \left(\gamma^{\text{ZERO}}_{\text{SO}(1;3)}\gamma^{\text{FIVE}}_{\text{SO}(1;3)}\right) \otimes \gamma^{\text{ZERO}}_{\text{SO}(1;3)} \tag{B1a}$$

$$\gamma^{\text{FIVE}}_{\text{SO}(2;6)} = \gamma^{\text{FIVE}}_{\text{SO}(1;3)} \otimes \gamma^{\text{FIVE}}_{\text{SO}(1;3)} \tag{B1b}$$

$$\gamma^{\mu}_{1\text{SO}(2;6)} = \gamma^{\mu}_{\text{SO}(1;3)} \otimes \gamma^{\text{FIVE}}_{\text{SO}(1;3)} \tag{B1c}$$

$$\gamma^{\mu}_{2\text{SO}(2;6)} = 1_{\text{SO}(1;3)} \otimes \gamma^{\mu}_{\text{SO}(1;3)} \ . \tag{B1d}$$

For clarity, we will write the decomposition of the matrices vertically. Branching rules for scalar can be derived by making use of the following diagram

$$\begin{array}{r|cccc} s(x,x') & \psi^\dagger(x,x') & \gamma^{\text{ZERO}} & \psi(x,x') \\ s_p(x) & \psi^\dagger(x) & \gamma^{\text{ZERO}}\gamma^{\text{FIVE}} & \psi(x) \\ s(x') & \psi^\dagger(x') & \gamma^{\text{ZERO}} & \psi(x') \end{array} \tag{B2}$$

Every element in the first line and right hand side of the diagram is the direct product of the corresponding elements from the second and third lines. The left hand side of the diagram shows the branching rule

$$s(x,x') \to s_p(x)s(x') \ , \quad s_p(x,x') \to s(x)s_p(x') \ . \tag{B3}$$

The branching rules for vector fields are computed similarly

$$\begin{array}{r|ccccc} \phi_1^\mu(x,x') & \psi^\dagger(x,x') & \gamma^{\text{ZERO}} & \gamma_1^\mu & \psi(x,x') \\ \phi_p^\mu(x) & \psi^\dagger(x) & \gamma^{\text{ZERO}}\gamma^{\text{FIVE}} & \gamma^\mu & \psi(x) \\ s_p(x') & \psi^\dagger(x') & \gamma^{\text{ZERO}} & \gamma^{\text{FIVE}} & \psi(x') \end{array} \tag{B4}$$

and the final results for vectors and pseudovectors are

$$\phi_1^\mu(x,x') \to \phi_p^\mu(x)s_p(x'), \tag{B5a}$$

$$\phi_{p1}^\mu(x,x') \to \phi^\mu(x)s(x'), \tag{B5b}$$

$$\phi_2^\mu(x,x') \to s_p(x)\phi^\mu(x'), \tag{B5c}$$

$$\phi_{p2}^\mu(x,x') \to s(x)\phi_p^\mu(x'), \tag{B5d}$$

where the branching of $\phi_{pj}^\mu(x,x')$ has been calculated by the same method.

We will calculate in detail some elements of 28- and 70-dimensional fields. The calculations below require sets of four different indexes satisfying one of following equations

$$\gamma^\mu\gamma^{\mu'}\gamma^\nu\gamma^{\nu'} = i\gamma^{\text{FIVE}}\epsilon^{\mu\mu'\nu\nu'} \ , \tag{B6a}$$

$$\gamma^\mu\gamma^{\mu'}\gamma^\nu = i\gamma^{\text{FIVE}}\epsilon^{\mu\mu'\nu\kappa}\gamma_\kappa \ , \tag{B6b}$$

$$\gamma^\mu\gamma^{\mu'} = -(i/2)\gamma^{\text{FIVE}}\epsilon^{\mu\mu'\lambda\lambda'}\gamma_\lambda\gamma_{\lambda'} \ . \tag{B6c}$$

Here, $\epsilon^{\mu\mu'\lambda\lambda'}$ is the totally antisymmetric four-index tensor. The branching of the tensor fields $\sigma_{jj'}^{\mu\mu'}(x,x')$ and $\sigma_{pjj'}^{\mu\mu'}(x,x')$ depend on the combination of jj' and must be carried out in three ways for 11, 22, and 12. For $jj'=11$ it is

$$\begin{array}{r|cccccc} \sigma_{11}^{\mu\mu'}(x,x') & \psi^\dagger(x,x') & \gamma^{\text{ZERO}} & \gamma_1^\mu & \gamma_1^{\mu'} & \psi(x,x') \\ \tilde{\sigma}^{\mu\mu'}(x) & \psi^\dagger(x) & \gamma^{\text{ZERO}}\gamma^{\text{FIVE}} & \gamma^\mu & \gamma^{\mu'} & \psi(x) \\ s(x') & \psi^\dagger(x') & \gamma^{\text{ZERO}} & \gamma^{\text{FIVE}} & \gamma^{\text{FIVE}} & \psi(x'), \end{array} \tag{B7a}$$

for $jj'=22$

$$\begin{array}{r|cccccc} \sigma_{22}^{\mu\mu'}(x,x') & \psi^\dagger(x,x') & \gamma^{\text{ZERO}} & \gamma_2^\mu & \gamma_2^{\mu'} & \psi(x,x') \\ s_p(x) & \psi^\dagger(x) & \gamma^{\text{ZERO}}\gamma^{\text{FIVE}} & & & \psi(x) \\ \sigma^{\mu\mu'}(x') & \psi^\dagger(x') & \gamma^{\text{ZERO}} & \gamma^\mu & \gamma^{\mu'} & \psi(x') \end{array} \tag{B7b}$$

The remaining terms have $jj' = 12, 21$

$$\begin{array}{r|cccccc} \sigma_{12}^{\mu\mu'}(x,x') & \psi^\dagger(x,x') & \gamma^{\text{ZERO}} & \gamma_1^\mu & \gamma_2^{\mu'} & \psi(x,x') \\ \phi_p^\mu(x) & \psi^\dagger(x) & \gamma^{\text{ZERO}}\gamma^{\text{FIVE}} & \gamma^\mu & & \psi(x) \\ \phi_p^{\mu'}(x') & \psi^\dagger(x') & \gamma^{\text{ZERO}} & \gamma^{\text{FIVE}} & \gamma^{\mu'} & \psi(x') \end{array} \tag{B7c}$$

The summary of the branching rules for all jj' is

$$\sigma_{11}^{\mu\mu'}(x,x') \to \tilde{\sigma}^{\mu\mu'}(x)s(x'), \quad \sigma_{p11}^{\mu\mu'}(x,x') \to \sigma^{\mu\mu'}(x)s_p(x'), \tag{B8a}$$

$$\sigma_{22}^{\mu\mu'}(x,x') \to s_p(x)\sigma^{\mu\mu'}(x'), \quad \sigma_{p22}^{\mu\mu'}(x,x') \to s(x)\tilde{\sigma}^{\mu\mu'}(x'), \tag{B8b}$$

$$\sigma_{12}^{\mu\mu'}(x,x') \to \phi_p^{\mu}(x)\phi_p^{\mu'}(x'), \quad \sigma_{p12}^{\mu\mu'}(x,x') \to \phi^{\mu}(x)\phi^{\mu'}(x'). \tag{B8c}$$

The branching for $jj' = 21$ is the same as $jj' = 12$, but with the opposite sign. Counts of degrees of freedom are $28 = 6\cdot 1 + 4_p \cdot 4_p + 1_p \cdot 6$ and $28_p = 6 \cdot 1_p + 4 \cdot 4 + 1 \cdot 6$.

A similar classification of branching rules by number occurrences of 1 and 2 in the index $jj'kk'$ is valid for $u_{jj'kk'}^{\mu\mu'\nu\nu'}(x,x')$. If all are 1, $jj'kk' = 1111$, then

$$\begin{array}{r|ccccccc} u_{1111}^{\mu\mu'\nu\nu'}(x,x') & \psi^{\dagger}(x,x') & \gamma^{\mathrm{ZERO}} & \gamma_1^{\mu} & \gamma_1^{\mu'} & \gamma_1^{\nu} & \gamma_1^{\nu'} & \psi(x,x') \\ i\epsilon^{\mu\mu'\nu\nu'}s(x) & \psi^{\dagger}(x) & \gamma^{\mathrm{ZERO}}\gamma^{\mathrm{FIVE}} & \gamma^{\mu} & \gamma^{\mu'} & \gamma^{\nu} & \gamma^{\nu'} & \psi(x) \\ s(x') & \psi^{\dagger}(x') & \gamma^{\mathrm{ZERO}} & \gamma^{\mathrm{FIVE}} & \gamma^{\mathrm{FIVE}} & \gamma^{\mathrm{FIVE}} & \gamma^{\mathrm{FIVE}} & \psi(x') \end{array} \tag{B9a}$$

The opposite case $jj'kk' = 2222$ is

$$\begin{array}{r|ccccccc} u_{2222}^{\mu\mu'\nu\nu'}(x,x') & \psi^{\dagger}(x,x') & \gamma^{\mathrm{ZERO}} & \gamma_2^{\mu} & \gamma_2^{\mu'} & \gamma_2^{\nu} & \gamma_2^{\nu'} & \psi(x,x') \\ s_p(x) & \psi^{\dagger}(x) & \gamma^{\mathrm{ZERO}}\gamma^{\mathrm{FIVE}} & & & & & \psi(x) \\ i\epsilon^{\mu\mu'\nu\nu'}s_p(x') & \psi^{\dagger}(x') & \gamma^{\mathrm{ZERO}} & \gamma^{\mu} & \gamma^{\mu'} & \gamma^{\nu} & \gamma^{\nu'} & \psi(x') \end{array} \tag{B9b}$$

Elements with three 1 and one 2, for example $jj'kk' = 1112$, are

$$\begin{array}{r|ccccccc} u_{1112}^{\mu\mu'\nu\nu'}(x,x') & \psi^{\dagger}(x,x') & \gamma^{\mathrm{ZERO}} & \gamma_1^{\mu} & \gamma_1^{\mu'} & \gamma_1^{\nu} & \gamma_2^{\nu'} & \psi(x,x') \\ i\epsilon^{\mu\mu'\nu\kappa}\phi_{\kappa}(x) & \psi^{\dagger}(x) & \gamma^{\mathrm{ZERO}}\gamma^{\mathrm{FIVE}} & \gamma^{\mu} & \gamma^{\mu'} & \gamma^{\nu} & & \psi(x) \\ \phi_p^{\nu'}(x') & \psi^{\dagger}(x') & \gamma^{\mathrm{ZERO}} & \gamma^{\mathrm{FIVE}} & \gamma^{\mathrm{FIVE}} & \gamma^{\mathrm{FIVE}} & \gamma^{\nu'} & \psi(x') \end{array} \tag{B9c}$$

Elements with one 1 and three 2, for example $jj'kk' = 1222$, are

$$\begin{array}{r|ccccccc} u_{1222}^{\mu\mu'\nu\nu'}(x,x') & \psi^{\dagger}(x,x') & \gamma^{\mathrm{ZERO}} & \gamma_1^{\mu} & \gamma_2^{\mu'} & \gamma_2^{\nu} & \gamma_2^{\nu'} & \psi(x,x') \\ \phi_p^{\mu}(x) & \psi^{\dagger}(x) & \gamma^{\mathrm{ZERO}}\gamma^{\mathrm{FIVE}} & \gamma^{\mu} & & & & \psi(x) \\ i\epsilon^{\mu'\nu\nu'\kappa'}\phi_{\kappa'}(x) & \psi^{\dagger}(x') & \gamma^{\mathrm{ZERO}} & \gamma^{\mathrm{FIVE}} & \gamma^{\mu'} & \gamma^{\nu} & \gamma^{\nu'} & \psi(x') \end{array} \tag{B9d}$$

The last group of elements of the four-index $SO(2;6)$ field has two 1 and two 2, for example $jj'kk' = 1122$,

$$\begin{array}{r|ccccccc} u_{1122}^{\mu\mu'\nu\nu'}(x,x') & \psi^{\dagger}(x,x') & \gamma^{\mathrm{ZERO}} & \gamma_1^{\mu} & \gamma_1^{\mu'} & \gamma_2^{\nu} & \gamma_2^{\nu'} & \psi(x,x') \\ \tilde{\sigma}^{\mu\mu'}(x) & \psi^{\dagger}(x) & \gamma^{\mathrm{ZERO}}\gamma^{\mathrm{FIVE}} & \gamma^{\mu} & \gamma^{\mu'} & & & \psi(x) \\ \sigma^{\nu\nu'}(x') & \psi^{\dagger}(x') & \gamma^{\mathrm{ZERO}} & \gamma^{\mathrm{FIVE}} & \gamma^{\mathrm{FIVE}} & \gamma^{\nu} & \gamma^{\nu'} & \psi(x') \end{array} \tag{B9e}$$

The branching rules may be summarized as follows.

$$u_{1111}^{\mu\mu'\nu\nu'}(x,x') \to i\epsilon^{\mu\mu'\nu\nu'}s(x)s(x')\,, \tag{B10a}$$

$$u_{2222}^{\mu\mu'\nu\nu'}(x,x') \to i\epsilon^{\mu\mu'\nu\nu'}s_p(x)s_p(x')\,, \tag{B10b}$$

$$u_{1112}^{\mu\mu'\nu\nu'}(x,x') \to i\epsilon^{\mu\mu'\nu\kappa}\phi_{\kappa}(x)\phi_p^{\nu'}(x')\,. \tag{B10c}$$

$$u_{1222}^{\mu\mu'\nu\nu'}(x,x') \to \phi_p^{\mu}(x)i\epsilon^{\mu'\nu\nu'\kappa'}\phi_{\kappa'}(x'). \tag{B10d}$$

$$u_{1122}^{\mu\mu'\nu\nu'}(x,x') \to \tilde{\sigma}^{\mu\mu'}(x)\sigma^{\nu\nu'}(x') \tag{B10e}$$

The count of degrees of freedom is $70 = 1\cdot 1 + 1_p \cdot 1_p + 4 \cdot 4_p + 4_p \cdot 4 + 6 \cdot 6$. The total antisymmetric tensor $\epsilon^{\mu\mu'\nu\nu'}$ will be omitted in the exchange rotation rules if the same indexes are one both sides of an equation.

[1] W. Pauli, Phys. Rev. **58** 716 (1940)

[2] I. Duck and E. C. G. Sudarshan, Am. J. Phys. **66** 284 (1998)

[3] E. H. Lieb, Rev. Mod. Phys. **48**, 553 (1976)

[4] W. Pauli *Exclusion Principle and Quantum Mechanics, Nobel Lectures, Physics 1942-1962*, Elsevier, Amsterdam (1964)
[5] R. P. Feynman, *The reason for antiparticles, The 1986 Dirac Memorial Lectures ed R P Feynman and S Weinberg*, Cambridge University Press, New York (1940)
[6] M. V. Berry, Nonlinearity **21** T19 (2008)
[7] M. V. Berry and J. M. Robbins, Proc. Roy. Soc. Land. **A453** 1771 (1997)
[8] M. V. Berry and J. M. Robbins, J. Phys. A: Math. Gen., **33** 207 (2000)
[9] M Peshkin, Phys. Rev. A **67** 042102 (2003)
[10] P. O′Hara arXiv:quant-ph/0310016 (2003)
[11] C. Wittig, J. Phys. Chem. A, **113** 7244 (2009)
[12] A. F. Reyes-Lega J. Phys. A **44** 325308 (2011)
[13] A. Jabs arXiv:0810.2399v4 [quant-ph] (2014)
[14] R. W. Hartung *Am. J. Phys.* **47** 900 (1979)
[15] J. M. Harrison and J. M. Robbins *J. Math. Phys.* **45** 1332 (2004)
[16] V. B. Berestetskii, L. P. Pitaevskii, and E M Lifshitz *Quantum Electrodynamics: Volume 4 (Course of Theoretical Physics)*, Butterworth-Heinemann, Oxford (1982)
[17] R. Slansky *Phys. Rep.* **79** 1 (1981)