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Branching rules for superalgebras with $so(N, 3N)$ even part and scalar odd part

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Odd N special orthogonal Lie algebra $so(N; 3N)$ together with scalar supercharges Q, Q^* give rise to a superalgebra where supermultiplets are made from conjugated representations of $so(N; 3N)$. We report branching rules for this superalgebra and consistency with the theory of exchange rotations.

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The purpose of this work is to analyze superalgebras[1] obtained by making use of an orthogonal Lie algebra and a scalar supercharge. For the first glance this will lead to same symmetry (spin) of both commuting and anticommuting fields, as in the theory of disordered metals.[2] In this example commuting fields are replica of anticommuting fields leading to few subalgebras of $\mathfrak{gl}(n|n)$.[3] The exception however is the case when supercharge also acts as the charge conjugation. In this case anticommuting and commuting fields have same symmetry (spin) but will represent physically different particles.

The spatial rotations for spinor representations of $so(N; 3N)$ algebra with odd N are defined as

$$\delta\psi = T\psi \ , \quad \delta\psi^c = T\psi^c \ , \tag{1}$$

and the superrotations are generated by the Grassmannian supercharge Q satisfying $QQ^c = -Q^cQ$

$$\begin{aligned} \delta'\psi &= Q^c\psi^c \ , \quad \delta'\psi^c = Q\psi, \\ \delta'\psi^t &= \psi^{ct}Q^c \ , \quad \delta'\psi^{ct} = \psi^t Q \ , \end{aligned} \tag{2}$$

Here ψ is $so(N; 3N)$ bispinor $\psi_L \oplus \psi_R$ and $\psi^c = C\psi^*$ is the charge conjugated bispinor $\psi_R^* \oplus \psi_L^*$. Fields ψ_L, ψ_R denote left and right spinors (dotted and undotted spinors in other notations). Explicitly the charge conjugation C depends on the choice of the representation for T in Eq. (1). Besides $Q^c = CQC$, and superscript t means transposed vector.

The key point of the present theory is that ψ_R^* has same space rotation rules as ψ_L and therefore can form supermultiplet with scalar supercharge

$$\begin{aligned} [\![\psi_\alpha, \psi'_\beta]\!] = 0 \ \Leftrightarrow \ \{\psi_L, \psi'_L\} = 0 \ , \quad [\psi_R, \psi'_R] = 0 \ , \\ [\psi_L, \psi'_R] = 0 \ , \quad [\![\psi^*_\alpha, \psi^{*\prime}_\beta]\!] = 0 \ . \end{aligned} \tag{3}$$

Fields ψ_R, ψ_R^* in this multiplet are commuting components and fields ψ_L, ψ_L^* are anticommuting components.

The superalgebra is constructed by making use of second variation in Eqs. (1,2). The grading is defined

$$\begin{aligned} &L_0 = \mathrm{End}(T) \quad L_{-1} = \mathrm{End}(Q^c) \quad L_1 = \mathrm{End}(Q) \\ &sso(N; 3N) = L_{-1} \oplus L_0 \oplus L_1 \ . \end{aligned} \tag{4}$$

Supercharges Q and Q^c must anticommute to an ideal of L_0 which is zero for the case of L_0 being the special orthogonal algebra.[4]

It is possible to introduce the maximal subalgebra $so(N; 3N) \otimes so(M; 3M)$ of the algebra $so(N + M; 3N + 3M)$, and branching rules for subalgebra embedding

$$so(N + M; 3N + 3M) \to so(N; 3N) \otimes so(M; 3M) \tag{5}$$

are as follows for scalar s, vector v, and adjoint ad representations

$$\begin{aligned} s^{N+M} = s^N \otimes s^M \quad v^{N+M} = v^N \oplus v^M \\ ad^{N+M} = ad^N \oplus ad^M \oplus v^N \otimes v^M \ , \end{aligned} \tag{6}$$

and for spinor fields

$$\begin{aligned} \psi_L^{N+M} &= \psi_L^N \otimes \psi_R^M \oplus \psi_R^N \otimes \psi_L^M \\ \psi_R^{N+M} &= \psi_L^N \otimes \psi_L^M \oplus \psi_R^N \otimes \psi_R^M \ . \end{aligned} \tag{7}$$

It is therefore possible to define ψ_L as anticommuting field and ψ_R as commuting field for any N and these commutation properties will be preserved upon embedding of maximal subalgebra given by Eq. (5).

The supermultiplet is defined only for odd N. Indeed, the supercharge Q is the scalar and therefore branching rules for it, see Eq. (6), become

$$\begin{aligned} Q^{N+M} &= Q^N \otimes Q^M = 0 \quad \text{odd } N, M \\ Q^{N+M} &= Q^N \quad \text{odd } N, N + M \end{aligned} \tag{8}$$

The scalar supercharge must be zero for even N, and therefore it is not possible to define grading by making use of a scalar supercharge for $so(N; 3N)$ with even N.

The summary of above embedding rules for maximal subalgebras and maximal subsuperalgebras is

$$\begin{aligned} so(N + M; 3N + 3M) &= sso(N; 3N) \otimes sso(M; 3M) \\ sso(N + M; 3N' + 3M') &= sso(N; 3N) \otimes so(M'; 3M') \\ so(N' + M'; 3N' + 3M') &= so(N'; 3N') \otimes so(M'; 3M') \end{aligned} \tag{9}$$

where N, M are odd and N', M' are even. Branching rules for generators of space rotations T^{N+M} follow those of adjoint representation Eq. (6). Therefore subalgebras in Eq. (9) are not just direct product of algebras but a more complicated operation.

The superalgebra $sso(N;3N)$ defined by Eq. (3) have superscalar s and supervector v^μ_j similarly to the scalar and the vector of $so(N;3N)$. Both s and v^μ_j are invariant upon the superrotation and transform as scalas and vector upon space rotations

$$\delta' s = 0\ , \quad s = \phi^{ct}\psi - \phi^t\psi^c \tag{10}$$

$$\delta' v^\mu_j = 0\ , \quad v^\mu_j = \phi^t\gamma^\mu_j\psi - \phi^{ct}\gamma^\mu_j\psi^c \tag{11}$$

The notation for γ-matrices is taken from Ref. [5], besides $\gamma^\mu_j Q^c - Q\gamma^\mu_j = 0$. Branching rules for these representations upon embeddings Eq. (9) are the same as for regular algebras Eq. (6).

Let's consider the case when the symmetry is lowered down to $SO(1;3)$ and maximal subalgebra is written as

$$sso(N;3N) = \underbrace{sso(1;3)\otimes\ldots\otimes sso(1;3)}_{N\text{times},N\text{is odd}}$$

$$so(N';3N') = \underbrace{sso(1;3)\otimes\ldots\otimes sso(1;3)}_{N'\text{times},N'\text{is even}}\ . \tag{12}$$

Our next step is to apply the exchange rotation to $sso(N;3N)$ or $so(N';3N')$.

The exchange rotation in space with group symmetry $SO(N;3N)$ smoothly exchanges coordinates x^μ_i and x^μ_j of two subspaces, where $i,j = 1,\ldots,N$ enumerates subspaces and $\mu = 0,\ldots,3$ enumerates coordinates within a subspce. The exchange rotates bispinor $\psi^N = \psi^N_L \oplus \psi^N_R$

$$\psi^{N\prime}(\ldots x_i \ldots x_j \ldots) = E_{ij}\psi^N(\ldots x_j \ldots x_i \ldots)\ . \tag{13}$$

It turns out that components of ψ^N can be indexed according to components of $\psi^{(1)}$ obtained by embedding Eq. (12). For example

$$\psi^N_{L\cdots L} \sim \underbrace{\psi^{(1)}_L \otimes\cdots\otimes \psi^{(1)}_L}_{N\text{times}}\ , \tag{14}$$

then the result of exchange rotations[5] is

$$\begin{aligned}
E_{ij}\psi^N_{\ldots L\ldots L\ldots}(x) &= -\psi^N_{\ldots L\ldots L\ldots}(x')\\
E_{ij}\psi^N_{\ldots L\ldots R\ldots}(x) &= \psi^N_{\ldots R\ldots L\ldots}(x')\\
E_{ij}\psi^N_{\ldots R\ldots L\ldots}(x) &= \psi^N_{\ldots L\ldots R\ldots}(x')\\
E_{ij}\psi^N_{\ldots R\ldots R\ldots}(x) &= \psi^N_{\ldots R\ldots R\ldots}(x')
\end{aligned} \tag{15}$$

Therefore exchange rotations in $SO(N;3N)$ behave as if the bispinor ψ^N is made by direct product of $sso(1;3)$ supermultiplets $\psi^{(1)} = \psi^{(1)}_L \oplus \psi^{(1)}_R$ with commutation rules given by Eq. (3).

The beauty of Eq. (15) is in connection between rotations in $4N$ dimensions and fields commutation rules in 4 dimensions. The symmetry lowering from $4N$ dimensions to 4 dimensions transforms space rotations to the exchange of fields and obtained commutation rules must be supersymmetric. In this way the supersymmetry is naturally derived from space rotation in higher dimensions.

Supermultiplets in the superalgebra $sso(1;3)$ are made from fields having same spin and therefore violate the Pauli spin–statistics theorem. Let me mention that in this case one of foundations of the spin–statistics theorem is not fullfilled and therefore the theorem is not violated.

The Pauli spinstatistics theorem was derived assuming that "actual ensemble of several like particles" is never realized in Nature as "a mixture of" commuting and anticommuting fields.[6] This is because the thermodynamic potential can only be defined if an ensemble of several like particles has well defined statistics.

Mathematically the prove of the Pauli spin–statistics theorem is based on the assumption that the charge conjugation always connects fields with the same statistics (both commuting or both anticommuting). This foundation principle is stronger than just requirements of no mixture of commuting and anticommuting fields in an ensemble of several like particles. The charge conjugation can connect two ensembles of different particles. Then the thermodynamic potential can be defined for each ensemble separately and therefore these two ensembles can have different statistics.

The present theory discuss possibility of supersymmetry between different fields (still connected by the charge conjugation), and therefore violates foundations of the spin–statistics theorem. The spin–statistics theorem is not applicable to the case when the supersymmetry comes together with the charge conjugation.

In summary we introduced the special orthogonal superalgebra constructed from odd N special orthogonal algebra $so(N;3N)$ and the scalar supercharge. We derived branching rules for maximal super subalgebra. We discussed connection between exchange rotations in $4N$ dimensions and supersymmetry in 4 dimensions.

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